ODE/PDE Qualifying Examination. August 2008

Directions:

Please answer all **five** questions. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages in the order you want them to be read, and identify yourself in each page by writing **all the digits** of your Banner ID #. **Please do not write your name in the exam**.

Questions:

1. Consider the 2×2 ODE system $\begin{cases} x' = -x + y \\ y' = x^2 - y \end{cases}$.

   (a) Find all the equilibrium points and their corresponding linearized equations.
   (b) Classify each equilibrium point as to their stability and their nature (node, saddle, etc).

2. A set $\Omega \subset \mathbb{R}^2$ is called an invariant region (or trapping region) for the 2×2 ODE system $x' = f(x, y)$, $y' = g(x, y)$ iff whenever $(x(t_0), y(t_0)) \in \Omega$ then $(x(t), y(t)) \in \Omega$ for all $t \geq t_0$.

   (a) State the Poincaré–Bendixon theorem and use it to prove that if Ω is a bounded invariant region without critical points, then Ω contains a periodic solution of the ODE system.
   (b) Show that the second order ODE $x'' + p(x, x')x' + x = 0$, with $p(x, y) = x^4 + y^4 - 1$ has at least one periodic solution.

3. (a) Solve the initial-value-problem $u_t + x^2 u_x = u$, $u(x, 0) = x$, using the method of characteristics.
   (b) Does the "initial-value-problem" $u_t + x^2 u_x = u$, $u(0, t) = t$ have one solution, infinitely many solutions, or no solution? Please explain why.

4. (a) Find the solution of Laplace's equation on the unit disc $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ with Neumann boundary condition on $\partial D = \{(\cos\theta, \sin\theta) | 0 \leq \theta < 2\pi\}$:

$$\nabla^2 u = 0, \qquad \text{for } (x, y) \in D$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = f(\theta), \qquad \text{with } f(\theta) = \cos\theta + \sin 2\theta.$$

   (b) Can you solve the previuos problem if $f(\theta) = 1 + \cos\theta$? Please justify your answer.
   Hint. In polar coordinates $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.

5. (a) Find the solution of the heat equation $u_t = u_{xx}$, $-\infty < x < \infty$, $0 < t < \infty$, with initial data $u(x, 0) = H(x) = \begin{cases} 0, & \text{if } -\infty < x < 0 \\ 1/2, & \text{if } \quad x = 0 \\ 1, & \text{if } 0 < x < \infty \end{cases}$

   and show that $\lim_{t \to 0^+} u(x, t) = H(x)$.
   (b) Show that $0 \leq u(x, t) \leq 1$ for all (x, t), with $-\infty < x < \infty$, $0 \leq t < \infty$.
   (c) Find $\lim_{t \to \infty} u(x, t)$.