UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Spring 2009

Instructions: There are four (4) problems on this examination, worth 25 points each. Work all 4 problems.

1. Consider the Mathieu equation

$$\frac{d^2u}{dx^2} + (\delta + \epsilon \cos x)\, u = 0$$

   (a) Show that for any arbitrarily small neighborhood of the point $(n^2/4, 0)$, $n = 0, 1, 2, \ldots$ in the (δ, ϵ) (real) parameter plane one can find solutions that remain bounded as $x \to \pm\infty$ (stable solutions) as well as unbounded (unstable) solutions.

   (b) Find a small-ϵ description for the curve(s) through the point $(\delta, \epsilon) = (1/4, 0)$ $(n = 1)$ separating regions of stability from regions of instability. Assume small ϵ expansions of the form

$$\begin{aligned} \delta &= \frac{1}{4} + \epsilon\delta_1 + \mathcal{O}\left(\epsilon^2\right) \\ u(x\,;\delta,\epsilon) &= u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \mathcal{O}\left(\epsilon^3\right) \ . \end{aligned}$$

   Determine δ_1.

2. Let $A(t)$ be a continuous real valued square matrix.

   (a) Show that every solution of the non autonomous linear system

$$\mathbf{x}' = A(t)\mathbf{x}$$

   satisfies ($|.|$ and $||.||$ are the vector and matrix 2-norms, resp.)

$$|\mathbf{x}(t)| \leq |\mathbf{x}(0)| \exp\left[\int_0^t ||A(s)||ds\right] \ .$$

   Hint: use the inequality

$$|\mathbf{x}(t)| \leq |\mathbf{x}(0)| + \int_0^t ||A(s)|||\mathbf{x}(s)|ds$$

   (show this!) to arrive at an extension of the Gronwall lemma.

(b) Show that if $\int_0^\infty ||A(s)||ds < \infty$ then every solution of the system in part (1) has a finite limit as t approaches infinity.
Hint: Show that part (1) implies that $|\mathbf{x}(t)| < M$ where M is some positive constant. Then show that

$$|\mathbf{x}(t_n) - \mathbf{x}(\hat{t}_n)| \leq \left| \int_{\hat{t}_n}^{t_n} ||A(s)|||\mathbf{x}(s)|ds \right| \leq M \left| \int_{\hat{t}_n}^{t_n} ||A(s)||ds \right|$$

so that, if t_n and $\hat{t}_n$ are two sequences approaching infinity, and $T_n = min(t_n, \hat{t}_n)$ then

$$|\mathbf{x}(t_n) - \mathbf{x}(\hat{t}_n)| \leq M \int_{T_n}^{\infty} ||A(s)||ds \to 0$$

as $n \to \infty$ (why?). Use this to conclude that $\mathbf{x}(t)$ must have a unique limit point as $t \to \infty$ over any sequence.

3. Solve

$$\frac{\partial \varphi}{\partial t} + c_0 \frac{\partial \varphi}{\partial x} + \varphi^2 = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$\varphi(x,t)|_{t=0} = f(x), \quad -\infty < x < \infty$$

and c_0 is a positive constant.

4. Find the disturbance due to a harmonic point source in an infinitely long acoustic waveguide with hard walls. Thus, solve

$$\begin{aligned} \triangle u + k^2 u = \delta(x-\xi)\delta(y-\eta) \quad &, \quad 0 < x < \infty \ , \ 0 < y < b \\ u \ \text{ outgoing at } x = \pm\infty \quad &, \quad u = 0 \ \text{ on } \ y = 0 \ \text{ and } y = b \ . \end{aligned}$$

Here we are solving

$$v_{xx} + v_{yy} - \frac{1}{c^2} v_{tt} = \delta(x-\xi)\delta(y-\eta)e^{i\omega t}$$

with

$$v(x,y,t) = u(x,y)e^{i\omega t} \ , \ k = \frac{\omega}{c} \ .$$

Hint: let

$$u(x,y) = \sum_1^\infty u_n(x) \sin \frac{n\pi y}{b} \ ;$$

multiply d.e. by $\sin(n\pi y/b)$ and integrate over y from 0 to b. Then solve the resulting problem for the $u_n(x)$ either by Fourier transform or by constructing Green's function directly.