**ODE/PDE Qualifying Exams, Fall 2009**

*Instructions:* Complete all problems. Start each problem on a new sheet of paper. Use only one side of each sheet of paper. Number the pages in the order you want them to be read. Identify yourself by writing your Banner ID # on each page.

Please do not write your name on the exam papers.

1) Consider the system $x' = y,\ y' = -x + x^3$.

a) Find all equilibria and linearize about the one at $(1,0)$. Find the general solution of the linearized system by finding the associated eigenvalues and eigenvectors and sketch the phase plane portrait for the linearized system.

b) Show that $E(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$ is a constant of the motion. Use this conservation law to show that the stable and unstable manifolds for $(1,0)$ are given by $y = \pm\frac{1}{\sqrt{2}}(x^2-1)$. Sketch these manifolds, discuss the behavior of solutions on them and relate them to the linearization in (a).

c) Solve the IVP $x' = \frac{1}{\sqrt{2}}(x^2-1),\ x(0) = \sqrt{2}$ and then determine the maximal interval of existence and the behavior as $t$ approaches the end points of the interval. Is there blow up in finite time? Discuss your result in the context of your sketch in part b).

Hint: Use partial fractions or the fact that an antiderivative of $(x^2-1)^{-1}$ is $\frac{1}{2}\log\Big((x-1)/(x+1)\Big)$.

2) Consider the IVP

$$x'' + x - \epsilon x^3 = 0,\ x(0) = A,\ x'(0) = 0, \tag{0.1}$$

where $\epsilon$ is small and positive.

a) Sketch the phase plane portrait and use it to determine the range of $A$ such that solutions are periodic.

b) Determine the first two terms in the regular perturbation expansion, $x = x_0 + \epsilon x_1 + \cdots$. Identify the non-periodic term in $x_1$.

c) The previous two items point to an apparent paradox. What is the paradox and what is its resolution?

Hint for b): You may use that

$$\cos^3(t) = \frac{1}{8}\cos(3t) - \frac{3}{8}\cos(t)\ .$$

3) Solve the following two initial value problems:
a)

$$u_t + u_x = u, \quad u(x,0) = \sin x \ ,$$

b)

$$u_t + u_x = u^2, \quad u(x,0) = \sin x \ .$$

For each solution, state in which maximal time interval $-T < t < T$ the solution is a smooth function defined for

$$-\infty < x < \infty, \quad -T < t < T \ .$$

4) Let $f : \mathbb{R} \to \mathbb{R}$ denote a smooth integrable function and set

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(y)| \, dy \ .$$

Let $u(x,t)$ denote the heat kernel solution of

$$u_t = u_{xx}, \quad u(x,0) = f(x) \ ,$$

i.e.,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y) \, dy \ .$$

Prove the following decay estimates:
a)

$$\sup_x |u(x,t)| \le \frac{\|f\|_1}{\sqrt{4\pi t}}, \quad t > 0 \ ;$$

b)

$$\sup_x |u_x(x,t)| \le \frac{C\|f\|_1}{t}, \quad t > 0 \ ,$$

where $C$ is a constant independent of $t$ and $f$. (To prove this, you may differentiate the formula for $u$ under the integral sign.)

c) Make a conjecture for a corresponding estimate of

$$\sup_x |u_{xx}(x,t)| \ .$$

What is the right power of $t$?