UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Fall 2010

Instructions: There are six (6) problems on this examination. Work all problems.

1. (15 points) Consider the following ODE system $\begin{cases} x' &= -x^2y+y-y^3 \\ y' &= -x^3+x^5+x^3y^2 \end{cases}$.

   (a) Find all equilibrium points.

   (b) Find linearized systems at points $(0,0)$, $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$.

   (c) Determine the type and stability of each of these 3 points for the respective linearized systems.

2. (25 points) Consider the following initial-value-problem (IVP)

$$yy'' + (y')^2 + f(y) = 0, \quad y(t)|_{t=0} > 0, \quad y'(t)|_{t=0} > 0, \quad t \geq 0,$$

   where $f(y)$ is a continuous function such that $f(y) \leq -f_0 y^{2+\epsilon}$, $f_0 > 0$ and $\epsilon > 0$, for all y. Hint: use a change of variable to reduce the order of the equation.

   (a) Prove that the solution of the IVP blows up in finite time (i.e. $|y(t)| \to \infty$ for $t \to T < \infty$).

   (b) Find estimate for the blow up time T.

   (c) Rewrite that IVP (by a change of variable) as a conservative system in a potential.

3. (15 points) Find similarity solutions to the problem

$$u_t = u_{xx}, \quad t > 0, x > 0; \quad u(x,0) = 0, \quad x > 0; \quad u(0,t) = 1, \quad t > 0.$$

   Namely, look for a solution $U(x,t) = f(x/\sqrt{t})$ and reduce the problem to a boundary value problem for an ordinary differential equation for $f(z)$, where z is the similarity variable.

4. (15 points) Solve the problem

$$xu_x + u_y = 1, \quad x \in \mathbb{R}, \quad y > 0, \quad u(x,0) = \exp(x).$$

5. (15 points) Let $U \subset \mathbb{R}^n$ be an open set. Show that a function $v \in C^2(U)$ that satisfies the mean-value property

$$v(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} v \, dS$$

   for each closed ball $B_r(x)$ of radius r centered at x with $B_r(x) \subset U$ is necessarily harmonic. In the above, $\omega_n = n\pi^{n/2}/\Gamma(\frac{n}{2}+1)$ is the surface area of the unit sphere in $\mathbb{R}^n$.

6. (15 points) Show that the Cauchy problem for Laplace's equation is ill-posed. Consider the problem in two dimensions

$$\Delta u = 0, \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x,0) = f(x), \quad u_y(x,0) = g(x), \quad -\infty < x < \infty$$

Construct a sequence of separated solutions

$$u_n(x,y) = \frac{1}{n} Y_n(y) \cos nx$$

such that $u_n(x,0) \to 0$, $(u_n)_y(x,0) \to 0$ as $n \to \infty$, while $u_n(x,1) \to \infty$. How can this be used to show that the solution does not depend continuously on the initial data?