UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Spring 2011

Instructions: There are six (6) problems on this examination. Work all problems.

1. (15 points) Locate the equilibrium points of the following ODE system

$$\begin{cases} x' &= x(x^2+y^2-1) \\ y' &= y(x^2+y^2-1) \end{cases} .$$

Sketch the phase plane diagram, and discuss the stability properties of all equilibrium points.

2. (15 points) For the ODE system
$$\begin{cases} x' &= X(x,y) \\ y' &= Y(x,y) \end{cases}$$
show that there are no closed paths in a simply-connected region in which
$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y}$
is of one sign, where $\rho(x,y)$ is any function having continuous first partial derivatives.

3. (20 points) The ODE system
$$\begin{cases} x_1' &= (-\sin 2t)x_1 + (\cos 2t - 1)x_2 \\ x_2' &= (\cos 2t + 1)x_1 + (\sin 2t)x_2 \end{cases}$$
has a fundamental matrix of normal solutions:

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{pmatrix}.$$

Obtain a matrix E such that $\Phi(t+\pi) = \Phi(t)E$ and find the Floquet exponents.

4. (15 points) Consider the equation

$$yu_x + xu_y = xy^3$$

with the boundary conditions $u = x^2$ on $y = 0,\ 1 < x < 2$. In what region of (x,y) space is the solution determined? What is the solution?

5. (15 points) Show that the solution to the nonlinear equation

$$u_x + u_y = u^2$$

passing through the initial curve

$$x = t \ , \quad y = -t \ , \quad u = t \ ,$$

becomes infinite along the hyperbola $x^2 - y^2 = 4$.

6. (20 points) Consider a cylindrical waveguide of radius a and infinite length, with absorbing boundary conditions at the walls and a vibrating diaphragm at $z = 0$ oscillating at frequency ω. Find the general solution for waves outgoing at $z = \pm\infty$. That is, solve

$$u_{tt} = c^2 \triangle u + \delta(z)e^{i\omega t} \quad ,0 \le r < a \ , \quad 0 \le \theta < 2\pi \quad , -\infty < z < \infty \ ,$$

with $u(a, \theta, z, t) = 0$. Show that if $\omega < \omega_0$ there are no propagating wave solutions and find an expression for ω_0.

Hints: (a) Look for solution in the following form (you need to justify why it is possible to drop the dependence on θ):

$$u(r, \theta, z, t) = R(r)Z(z)e^{i\omega t}.$$

Carefully discuss the cases $z < 0$ and $z > 0$ and ensure that in each case the z-dependence leads to either outgoing waves or bounded behavior at infinity. Green's functions could be helpful here, but you can simply work away from $z = 0$ and impose the necessary conditions on the solution at $z = 0$ implied by the δ-function forcing to connect the expansions in the positive and negative half-line. Note that here we are only interested in the "particular" solution consistent with the forcing and BC, while the "outgoing-wave" condition implies that any homogeneous solution part must be set to zero.

(b) The solution of the following ODE

$$x^2 f'' + xf' + (x^2 - m^2)f = 0, \ m = 0, 1, 2, \dots,$$

which is nonsingular at $x = 0$, is given by the Bessel function $J_m(x)$. Let the n-th non-trivial zero of the Bessel function $J_m(x)$ be x_{mn}, i.e. $J_m(x_{mn}) = 0$ (assume that $x_{mn} > 0$). The smallest zero x_{01} of the Bessel function J_0 is given by $x_{01} \simeq 2.4048$ and x_{mn} grows with increasing m, n. You can assume that all zeros of $J_m(x)$ are known.

(c) The solution of the following ODE

$$x^2 f'' + xf' - (x^2 + n^2)f = 0, \ n = 0, 1, 2, \dots,$$

with no singularity at $x = 0$, has no zeros in $0 \le x < \infty$.

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August 2011

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1. (15 points) Locate the equilibrium points of the following ODE system

$$\begin{cases} x' &= x(x^2+y^2-1) \\ y' &= y(x^2+y^2-1) \end{cases} .$$

Sketch the phase plane diagram, and discuss the stability properties of all equilibrium points.

2. (15 points) For the ODE system
$$\begin{cases} x' &= X(x,y) \\ y' &= Y(x,y) \end{cases}$$
show that there are no closed paths in a simply-connected region in which
$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y}$
is of one sign, where $\rho(x,y)$ is any function having continuous first partial derivatives.

3. (20 points) The ODE system
$$\begin{cases} x_1' &= (-\sin 2t)x_1 + (\cos 2t - 1)x_2 \\ x_2' &= (\cos 2t + 1)x_1 + (\sin 2t)x_2 \end{cases}$$
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Obtain a matrix E such that $\Phi(t+\pi) = \Phi(t)E$ and find the Floquet exponents.

4. (15 points) Consider the equation

$$yu_x + xu_y = xy^3$$

with the boundary conditions $u = x^2$ on $y = 0$, $1 < x < 2$. In what region of (x,y) space is the solution determined? What is the solution?

5. (15 points) Show that the solution to the quasilinear equation

$$u_x + u_y = u^2$$

passing through the initial curve

$$x = t \ , \quad y = -t \ , \quad u = t \ ,$$

becomes infinite along the hyperbola $x^2 - y^2 = 4$.

6. (20 points) Consider a cylindrical waveguide of radius a and infinite length, with absorbing boundary conditions at the walls and a vibrating diaphragm at $z = 0$ oscillating at frequency ω. Find the general solution for waves outgoing at $z = \pm\infty$. That is, solve

$$u_{tt} = c^2 \Delta u + \delta(z) e^{i\omega t} \quad , 0 \leq r < a \quad , \quad 0 \leq \theta < 2\pi \quad , -\infty < z < \infty \; ,$$

with $u(a, \theta, z, t) = 0$. Show that if $\omega < \omega_0$ there are no propagating wave solutions and find an expression for ω_0.

Hints: (a) Look for solution in the following form (you need to justify why it is possible to drop the dependence on θ):

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Carefully discuss the cases $z < 0$ and $z > 0$ and ensure that in each case the z-dependence leads to either outgoing waves or bounded behavior at infinity. Green's functions could be helpful here, but you can simply work away from $z = 0$ and impose the necessary conditions on the solution at $z = 0$ implied by the δ-function forcing to connect the expansions in the positive and negative half-line. Note that here we are only interested in the "particular" solution consistent with the forcing and BC, while the "outgoing-wave" condition implies that any homogeneous solution part must be set to zero.

(b) The solution of the following ODE

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(c) The solution of the following ODE

$$x^2 f'' + x f' - (x^2 + n^2) f = 0, \;\; n = 0, 1, 2, \ldots,$$

with no singularity at $x = 0$, has no zeros in $0 \leq x < \infty$.

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Fall 2010

Instructions: There are six (6) problems on this examination. Work all problems.

1. (15 points) Consider the following ODE system $\begin{cases} x' &= -x^2y + y - y^3 \\ y' &= -x^3 + x^5 + x^3y^2 \end{cases}$.

   (a) Find all equilibrium points.

   (b) Find linearized systems at points $(0,0)$, $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$.

   (c) Determine the type and stability of each of these 3 points for the respective linearized systems.

2. (25 points) Consider the following initial-value-problem (IVP)

$$yy'' + (y')^2 + f(y) = 0, \quad y(t)|_{t=0} > 0, \quad y'(t)|_{t=0} > 0, \quad t \geq 0,$$

   where $f(y)$ is a continuous function such that $f(y) \leq -f_0 y^{2+\epsilon}$, $f_0 > 0$ and $\epsilon > 0$, for all y. Hint: use a change of variable to reduce the order of the equation.

   (a) Prove that the solution of the IVP blows up in finite time (i.e. $|y(t)| \to \infty$ for $t \to T < \infty$).

   (b) Find estimate for the blow up time T.

   (c) Rewrite that IVP (by a change of variable) as a conservative system in a potential.

3. (15 points) Find similarity solutions to the problem

$$u_t = u_{xx}, \quad t > 0, x > 0; \quad u(x,0) = 0, \quad x > 0; \quad u(0,t) = 1, \quad t > 0.$$

   Namely, look for a solution $U(x,t) = f(x/\sqrt{t})$ and reduce the problem to a boundary value problem for an ordinary differential equation for $f(z)$, where z is the similarity variable.

4. (15 points) Solve the problem

$$xu_x + u_y = 1, \quad x \in \mathbb{R}, \quad y > 0, \quad u(x,0) = \exp(x).$$

5. (15 points) Let $U \subset \mathbb{R}^n$ be an open set. Show that a function $v \in C^2(U)$ that satisfies the mean-value property

$$v(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} v \, dS$$

   for each closed ball $B_r(x)$ of radius r centered at x with $B_r(x) \subset U$ is necessarily harmonic. In the above, $\omega_n = n\pi^{n/2}/\Gamma(\frac{n}{2}+1)$ is the surface area of the unit sphere in $\mathbb{R}^n$.

6. (15 points) Show that the Cauchy problem for Laplace's equation is ill-posed. Consider the problem in two dimensions

$$\Delta u = 0, \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x,0) = f(x), \quad u_y(x,0) = g(x), \quad -\infty < x < \infty$$

Construct a sequence of separated solutions

$$u_n(x,y) = \frac{1}{n} Y_n(y) \cos nx$$

such that $u_n(x,0) \to 0$, $(u_n)_y(x,0) \to 0$ as $n \to \infty$, while $u_n(x,1) \to \infty$. How can this be used to show that the solution does not depend continuously on the initial data?