UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

August 2011

*Instructions:* There are six (6) problems on this examination. Work all problems.

1. (15 points) Locate the equilibrium points of the following ODE system

$$\begin{cases} x' &= x(x^2+y^2-1) \\ y' &= y(x^2+y^2-1) \end{cases} .$$

   Sketch the phase plane diagram, and discuss the stability properties of all equilibrium points.

2. (15 points) For the ODE system
$\begin{cases} x' &= X(x,y) \\ y' &= Y(x,y) \end{cases}$
show that there are no closed paths in a simply-connected region in which
$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y}$
is of one sign, where $\rho(x,y)$ is any function having continuous first partial derivatives.

3. (20 points) The ODE system
$\begin{cases} x_1' &= (-\sin 2t)x_1 + (\cos 2t - 1)x_2 \\ x_2' &= (\cos 2t + 1)x_1 + (\sin 2t)x_2 \end{cases}$
has a fundamental matrix of normal solutions:

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{pmatrix}.$$

   Obtain a matrix $E$ such that $\Phi(t+\pi) = \Phi(t)E$ and find the Floquet exponents.

4. (15 points) Consider the equation

$$yu_x + xu_y = xy^3$$

   with the boundary conditions $u = x^2$ on $y = 0,\ 1 < x < 2$. In what region of $(x,y)$ space is the solution determined? What is the solution?

5. (15 points) Show that the solution to the quasilinear equation

$$u_x + u_y = u^2$$

   passing through the initial curve

$$x = t \quad , \quad y = -t \quad , \quad u = t \quad ,$$

   becomes infinite along the hyperbola $x^2 - y^2 = 4$.

6. (20 points) Consider a cylindrical waveguide of radius $a$ and infinite length, with absorbing boundary conditions at the walls and a vibrating diaphragm at $z = 0$ oscillating at frequency $\omega$. Find the general solution for waves outgoing at $z = \pm\infty$. That is, solve

$$u_{tt} = c^2 \Delta u + \delta(z) e^{i\omega t} \quad , 0 \le r < a \quad , \quad 0 \le \theta < 2\pi \quad , -\infty < z < \infty \, ,$$

with $u(a, \theta, z, t) = 0$. Show that if $\omega < \omega_0$ there are no propagating wave solutions and find an expression for $\omega_0$.

Hints: (a) Look for solution in the following form (you need to justify why it is possible to drop the dependence on $\theta$):

$$u(r, \theta, z, t) = R(r) Z(z) e^{i\omega t}.$$

Carefully discuss the cases $z < 0$ and $z > 0$ and ensure that in each case the $z$-dependence leads to either outgoing waves or bounded behavior at infinity. Green's functions could be helpful here, but you can simply work away from $z = 0$ and impose the necessary conditions on the solution at $z = 0$ implied by the $\delta$-function forcing to connect the expansions in the positive and negative half-line. Note that here we are only interested in the "particular" solution consistent with the forcing and BC, while the "outgoing-wave" condition implies that any homogeneous solution part must be set to zero.

(b) The solution of the following ODE

$$x^2 f'' + x f' + (x^2 - m^2) f = 0, \; m = 0, 1, 2, \ldots,$$

which is nonsingular at $x = 0$, is given by the Bessel function $J_m(x)$. Let the $n$-th non-trivial zero of the Bessel function $J_m(x)$ be $x_{mn}$, i.e. $J_m(x_{mn}) = 0$ (assume that $x_{mn} > 0$). The smallest zero $x_{01}$ of the Bessel function $J_0$ is given by $x_{01} \simeq 2.4048$ and $x_{mn}$ grows with increasing $m, n$. You can assume that all zeros of $J_m(x)$ are known.

(c) The solution of the following ODE

$$x^2 f'' + x f' - (x^2 + n^2) f = 0, \; n = 0, 1, 2, \ldots,$$

with no singularity at $x = 0$, has no zeros in $0 \le x < \infty$.