ODE/PDE MS/PhD Qualifying Exam. January 2012

Directions:

Please answer all **five** questions. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages in the order you want them to be read, and identify yourself on each page by writing **the last four digits** of your Banner ID #. **Please do not write your name on the exam**.

Questions:

1. Consider the 2×2 ODE system

$$\frac{dx}{dt} = x(x-1) + y^2$$
$$\frac{dy}{dt} = x^2 + y(y-2)$$

(a) Verify that $(x, y) = (0, 0)$ is an equilibrium point and study its stability using the linearized equations.
(b) Give the definition of Lyapunov function and state Lyapunov's stability criterion. Verify that $V(x, y) = x^2 + y^2$ is a Lyapunov function around $(x, y) = (0, 0)$ for the ODE system under consideration and use it to study the stability of the equilibrium point $(x, y) = (0, 0)$.

2. (a) State Floquet's theorem for a periodic ODE system of the form $\frac{d\vec{x}}{dt} = A(t)\vec{x}$, where $\vec{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$ and $A(t)$ is T-periodic in time, i.e. $A(t+T) = A(t)$ for all $t \in \mathbb{R}$.
(b) The ODE system

$$\frac{dx}{dt} = (-\sin 2t)x + (\cos 2t - 1)y$$
$$\frac{dy}{dt} = (\cos 2t + 1)x + (\sin 2t)y$$

has the fundamental matrix solution

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{pmatrix}.$$

Obtain the matrix E such that $\Phi(t+\pi) = \Phi(t)E$ and find the Floquet exponents.

3. (a) Solve the Initial-Value-Problem (IVP) $u_t + x^2 u_x = 0$, where $-\infty < x < \infty$, $0 < t < \infty$, and with initial data

$$u(x, 0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < 1 \\ 0, & 1 < x < \infty \end{cases},$$

using the method of characteristics.
(b) Plot the characteristics in the (x, t)–plane and identify the region in the (x, t)–plane where $u(x, t) \equiv 0$.

4. Consider $u(x,y,t) = \dfrac{e^{-(x^2+y^2)/4t}}{4\pi t}$, for $(x,y) \in \mathbb{R}^2$ and $0 < t < \infty$.
(a) Show that $u_t = u_{xx} + u_{yy}$ on $(x,y) \in \mathbb{R}^2$, $0 < t < \infty$.
(b) Show that $\lim\limits_{t \to 0^+} u(x,y,t) = \delta(x,y)$ in distribution sense on $(x,y) \in \mathbb{R}^2$.

5. (a) The function $G(\vec{x}) = \dfrac{1}{4\pi|\vec{x}|}$, where $\vec{x} = (x,y,z)$ and $|\vec{x}| = \sqrt{x^2+y^2+z^2}$ is a fundamental solution of the Laplacian in $\mathbb{R}^3$. What does this mean?
(b) Use $G(\vec{x})$ from part (a) to write down an integral expression for the solution $u(x,y,z)$ of Poisson's equation

$$\Delta u = f(\vec{x}) = \begin{cases} 1 & |\vec{x}| \leq 1 \\ 0 & |\vec{x}| > 1 \end{cases}$$

with boundary condition $u(x,y,z) \to 0$ as $|\vec{x}| \to \infty$.
(c) Show that if $u(\vec{x})$ is the solution of part (b), then $u(\vec{x})$ can be written as $u(\vec{x}) = -\dfrac{1}{|\vec{x}|}\left\{\dfrac{V}{4\pi} + r(\vec{x})\right\}$, where V is the volume of the unit ball $B = \{\vec{x} \in \mathbb{R}^3 | |\vec{x}| \leq 1\}$ and $r(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$.