UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Spring 2013

*Instructions:* There are six (6) problems on this examination. Work all problems.

1. Locate the equilibrium points of the following ODE system

$$\begin{cases} x' &= -x + x^3 + xy^2, \\ y' &= -y + x^2 y + y^3; \end{cases} .$$

(a) Sketch the phase plane diagram.
(b) Discuss the stability properties of all equilibrium points.
(c) If the origin is a critical point, propose and use a Lyapunov function in order to investigate stability of the nonlinear (!) system in the vicinity of the origin.

2. Show that the nonlinear system

$$\begin{aligned} \dot{x}_1 &= 4x_1 + x_2 - 4x_1^3 - 4x_1 x_2, \\ \dot{x}_2 &= -x_1 + 4x_2 - 4x_2 x_1^2 - 4x_2^3, \\ \dot{x}_3 &= -6x_3 + 3, \end{aligned}$$

has a periodic orbit $\gamma(t) = (\cos t, -\sin t, 1/2)^T$.
Find the linearization of this system about $\gamma(t)$, $\dot{\mathbf{x}} = A(t)\mathbf{x}$. Show that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} e^{-8t}\cos t & \sin t & 0 \\ -e^{-8t}\sin t & \cos t & 0 \\ 0 & 0 & e^{-6t} \end{pmatrix}.$$

Find the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

3. Show that the nonlinear system

$$\begin{aligned} \dot{x} &= 4x - y - 4x^3 - 7xy^2, \\ \dot{y} &= x + 4y - 4x^2 y - 7y^3, \end{aligned}$$

(a) has at least one periodic orbit in the annulus $\sqrt{4/7} \leq r \leq 1$ (for now use the fact, that the only critical point is the origin);
(b) there is exactly one orbit inside the annulus;
(c) prove that there are no critical points but the origin (**Hint:** consider angle derivative in the polar coordinates).

4. Let $v$ satisfy

$$v_{tt} + av = c^2 v_{xx}, \qquad -\infty < x < \infty, \quad t > 0, \tag{1}$$
$$v(x,0) = 0 = v_t(x,0), \qquad -\infty < x < \infty, \tag{2}$$
$$\text{for each } t,\ v(\cdot,t) \text{ has compact support,} \tag{3}$$

where $a > 0$ and $c$ are constants. By using the energy functional

$$E(t) = \int_{-\infty}^{\infty} \left[ v_t(x,t)^2 + c^2 v_x(x,t)^2 + av(x,t)^2 \right] dx,$$

show that $v(x,t) = 0$ for $-\infty < x < \infty$, $t > 0$.

5. Consider the initial value problem

$$u_t + uu_x + u = 0 \qquad (0 < x < 1, t > 0) \tag{4}$$
$$u(x,0) = a \sin x \tag{5}$$

Find the characteristics curves explicitly. Show that if $a > 1$ then a global smooth solution does not exist. Also, find the time of breakdown, i.e., the maximal time of existence of the smooth solution.

**NOTE: The Implicit Function Theorem**
Let $U \in \mathbb{R}^n$, $V \in \mathbb{R}^m$ denote nonempty sets and let $\Phi : U \times V \to \mathbb{R}^m$ be a $C^p$ function, $p > 1$. Let $(u_0, v_0) \in U \times V$ and assume $\Phi(u_0, v_0) = 0$, $\det(\Phi_v(u_0, v_0)) \neq 0$. Then there exist open sets $U_0$, $V_0$, with $u_0 \in U_0 \subset U$, $v_0 \in V_0 \subset V$ so that the following holds. For all $u_0 \in U_0$, the equation $\Phi(u,v) = 0$ has a solution $v \in V_0$ which is unique in $V_0$. In addition, the function $u \mapsto v$, which assigns to each $u \in U_0$ the corresponding $v \in V_0$, is $C^p$.

6. Let $a, c > 0$ be positive constants. The telegrapher's equation

$$u_{tt} + au_t = c^2 u_{xx}, \qquad x \in (0,1), \quad t > 0,$$

represents a damped version of the wave equation. Consider the Dirichlet boundary value problem with

$$u(t,0) = u(t,1) = 0 \quad \text{and} \quad u(0,x) = f(x), u_t(0,x) = 0$$

Find all separable solutions to the telegrapher's equation that satisfy the boundary conditions. Write down the series solution for the initial boundary value problem. (You do not have to verify orthonormality of the elements of a standard Fourier series, the fact that they form a basis, etc. Just write a formula for $u$ in a series form and for the coefficients of the series.)

ODE/PDE MS/PhD Qualifying Exam. January 2012

**Directions:**

Please answer all **five** questions. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages in the order you want them to be read, and identify yourself on each page by writing **the last four digits** of your Banner ID #. **Please do not write your name on the exam.**

**Questions:**

1. Consider the $2 \times 2$ ODE system

$$\frac{dx}{dt} = x(x-1) + y^2$$
$$\frac{dy}{dt} = x^2 + y(y-2)$$

(a) Verify that $(x, y) = (0, 0)$ is an equilibrium point and study its stability using the linearized equations.
(b) Give the definition of Lyapunov function and state Lyapunov's stability criterion. Verify that $V(x, y) = x^2 + y^2$ is a Lyapunov function around $(x, y) = (0, 0)$ for the ODE system under consideration and use it to study the stability of the equilibrium point $(x, y) = (0, 0)$.

2. (a) State Floquet's theorem for a periodic ODE system of the form $\frac{d\vec{x}}{dt} = A(t)\vec{x}$, where $\vec{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n \times n}$ and $A(t)$ is $T$-periodic in time, i.e. $A(t+T) = A(t)$ for all $t \in \mathbb{R}$.
(b) The ODE system

$$\frac{dx}{dt} = (-\sin 2t)x + (\cos 2t - 1)y$$
$$\frac{dy}{dt} = (\cos 2t + 1)x + (\sin 2t)y$$

has the fundamental matrix solution

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{pmatrix}.$$

Obtain the matrix $E$ such that $\Phi(t + \pi) = \Phi(t)E$ and find the Floquet exponents.

3. (a) Solve the Initial-Value-Problem (IVP) $u_t + x^2 u_x = 0$, where $-\infty < x < \infty$, $0 < t < \infty$, and with initial data

$$u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < 1 \\ 0, & 1 < x < \infty \end{cases},$$

using the method of characteristics.
(b) Plot the characteristics in the $(x, t)$–plane and identify the region in the $(x, t)$–plane where $u(x, t) \equiv 0$.

4. Consider $u(x,y,t) = \dfrac{e^{-(x^2+y^2)/4t}}{4\pi t}$, for $(x,y) \in \mathbb{R}^2$ and $0 < t < \infty$.
(a) Show that $u_t = u_{xx} + u_{yy}$ on $(x,y) \in \mathbb{R}^2$, $0 < t < \infty$.
(b) Show that $\lim\limits_{t \to 0+} u(x,y,t) = \delta(x,y)$ in distribution sense on $(x,y) \in \mathbb{R}^2$.

5. (a) The function $G(\vec{x}) = \dfrac{1}{4\pi|\vec{x}|}$, where $\vec{x} = (x,y,z)$ and $|\vec{x}| = \sqrt{x^2+y^2+z^2}$ is a fundamental solution of the Laplacian in $\mathbb{R}^3$. What does this mean?
(b) Use $G(\vec{x})$ from part (a) to write down an integral expression for the solution $u(x,y,z)$ of Poisson's equation

$$\Delta u = f(\vec{x}) = \begin{cases} 1 & |\vec{x}| \le 1 \\ 0 & |\vec{x}| > 1 \end{cases}$$

with boundary condition $u(x,y,z) \to 0$ as $|\vec{x}| \to \infty$.
(c) Show that if $u(\vec{x})$ is the solution of part (b), then $u(\vec{x})$ can be written as $u(\vec{x}) = -\dfrac{1}{|\vec{x}|}\left\{\dfrac{V}{4\pi} + r(\vec{x})\right\}$, where $V$ is the volume of the unit ball $B = \{\vec{x} \in \mathbb{R}^3 \mid |\vec{x}| \le 1\}$ and $r(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$.

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Fall 2012

*Instructions:* There are six (6) problems on this examination. Work all problems.

1. (10 points) Consider the following ODE system $\begin{cases} \dot{x}_1 &= (-1+x_1^2+x_2^2)x_2^5 \\ \dot{x}_2 &= x_1 - x_1^3 - x_1 x_2^2 \end{cases}$.

   (a) Find all equilibrium points.

   (b) Find linearized systems at points $(0,0)$, $(0,1)$ and $(2^{-1/2}, -2^{-1/2})$.

   (c) Determine the type and stability of each of these 3 points for the respective linearized systems.

2. (20 points) Use the Poincaré-Bendixson theorem and the fact that the planar system

$$\dot{x} = x + y - x^5 \ , \ \dot{y} = -x + y - y^5$$

   has only one critical point at the origin to show that this system has a periodic orbit in the annular region

$$A = \left\{ \mathrm{x} \in \mathbb{R}^2 \middle| 1 \leq |\mathrm{x}| \leq \sqrt{2} \right\}, \quad \mathrm{x} = (x, y).$$

   **Hint:** Convert to polar coordinates.

3. (15 points) Show that the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_1^2 x_2 - x_2^3 \\ \dot{x}_3 &= 2x_3 + 6 \end{aligned}$$

   has a periodic orbit $\gamma(t) = (\sin 2t, \cos 2t, -3)^T$. Find the linearization of this system about $\gamma(t)$, $\dot{\mathrm{x}} = A(t)\mathrm{x}$. Show that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} \cos 2t & e^{-2t} \sin 2t & 0 \\ -\sin 2t & e^{-2t} \cos 2t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

   Find the characteristic exponents and multipliers of $\gamma(t)$. State Floquet's theorem. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

4. (20 points) Solve the initial-value problem

$$tu_t + (t^4 + x)u_x = u^2, \quad t > 0, \quad x \in \mathbb{R}, \quad u(t,x)|_{t=2} = x + 4.$$

5. (20points) Use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

$$u_t - u_{xx} = a, \ x \in (0,l), \ t \in (0,\infty), \ l > 0, \ a = const,$$

$$u|_{x=0} = u|_{x=l} = 0, \ t \in (0,\infty),$$

$$u(x,t)|_{t=0} = \frac{a}{2}x(l-x) + \sin\left(\frac{3\pi x}{l}\right).$$

Hint: represent solution as $u = v + w$, where $v$ is the solution of nonhomogeneous $t$-independent problem and $w$ is the solution of $t$-dependent homogeneous (i.e. with zero right hand side) problem.

6. (15points) Prove the uniqueness of the solution $u(\mathbf{x},t) \in C_1^2(\mathbb{R}^2 \times [0,\infty))$ of the following initial/boundary value problem

$$u_t - \Delta u = 0, \ \mathbf{x} \in \Omega, \ t \in (0,\infty),$$

$$\mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0, \quad u(\mathbf{x},t)|_{t=0} = u_0(\mathbf{x}),$$

where $\Omega \in \mathbb{R}^2$ is the simply connected bounded open set with the smooth boundary $\partial\Omega$ and $\mathbf{n}$ is the unit normal vector to $\partial\Omega$.