DIFFERENTIAL EQUATIONS QUALIFYING EXAM FALL 2013

Tue, Aug. 13, 9:00 – 12:00 Department of Mathematics and Statistics University of New Mexico

Instructions: Complete all six (6) problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Locate the equilibrium points of the following ODE system

$$\begin{cases} x' = -4x + x^3 + xy^2, \\ y' = -4y + x^2y + y^3; \end{cases}$$

(a) Sketch the phase plane diagram.

(b) Discuss the stability properties of all equilibrium points.

(c) If the origin is a critical point, propose and use a Lyapunov function in order to investigate stability of the nonlinear (!) system in the vicinity of the origin.

2) Show that the nonlinear system

$$\dot{x}_1 = 4x_1 + 2x_2 - x_1^3 - x_1x_2^2, \dot{x}_2 = -2x_1 + 4x_2 - x_2x_1^2 - x_2^3, \dot{x}_3 = -3x_3 - 6,$$

has a periodic orbit $\gamma(t) = (2\cos(2t), -2\sin(2t), -2)^T$. Find the linearization of this system about $\gamma(t)$, $\dot{\mathbf{x}} = A(t)\mathbf{x}$. Show that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} e^{-8t} \cos 2t & \sin 2t & 0\\ -e^{-8t} \sin 2t & \cos 2t & 0\\ 0 & 0 & e^{-3t} \end{pmatrix}.$$

Find the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

3) Show that the nonlinear system

$$\dot{x} = 3x - y - 4x^3 - 7xy^2, \dot{y} = x + 3y - 4x^2y - 7y^3,$$

(a) has at least one periodic orbit in the annulus $\sqrt{3/7} \le r \le \sqrt{3}/2$ (for now use the fact, that the only critical point is the origin);

(b) there is exactly one orbit inside the annulus;

(c) prove that there are no critical points but the origin (**Hint:** consider angle derivative in the polar coordinates).

4) a) Determine solutions of the following two PDEs

II)

I)

$$u_t = u_{xxx} - u_{xx}, \quad x \in \mathbb{R}, \quad t \ge 0$$

11)

$$v_t = v_{xxx} + iv_{xx}, \quad x \in \mathbb{R}, \quad t \ge 0 ,$$

with initial condition

$$u(x,0) = v(x,0) = e^{ikx}$$

where k is real.

b) For which of the two PDEs is the Cauchy problem well–posed? For which is it ill–posed? Justify your answer.

5) a) Solve the PDE

$$u_t + u_x = \frac{1}{u}$$

with initial condition

$$u(x,0) = 5 + \sin x$$
 for $x \in \mathbb{R}$.

b) Does the solution exist for all real t (positive and negative)? If not, determine the maximal interval of existence.

6) Let

$$K(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}, \quad x \in \mathbb{R}, \quad t > 0,$$

denote a multiple of the 1D heat kernel.

a) For every fixed $x \in \mathbb{R}$ determine the limit

$$\lim_{t \to 0+} K_x(x,t)$$

where

$$K_x(x,t) = \frac{\partial K(x,t)}{\partial x} .$$

b) Is the function

$$u(x,t) = \begin{cases} K_x(x,t) & \text{for } x \in \mathbb{R}, \quad t > 0\\ 0 & \text{for } x \in \mathbb{R}, \quad t = 0 \end{cases}$$

continuous? Justify your answer.