

UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Spring 2014

Instructions: There are five (5) problems on this examination. Work all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

1. (15 points) Locate the equilibrium points of the following ODE system

$$\begin{cases} x' &= -4x + x^3 + xy^2, \\ y' &= -4y + x^2y + y^3; \end{cases} \quad .$$

- (a) (5 points) Sketch the phase plane diagram.
(b) (5 points) Discuss the stability properties of all equilibrium points.
(c) (5 points) If the origin is a critical point, propose and use a Lyapunov function in order to investigate stability of the nonlinear (!) system in the vicinity of the origin.
2. (20 points) Show that the nonlinear system

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 2x_2 - x_1^3 - x_1x_2^2, \\ \dot{x}_2 &= -2x_1 + 3x_2 - x_2x_1^2 - x_2^3, \\ \dot{x}_3 &= x_3 + 1, \end{aligned}$$

has a periodic orbit $\gamma(t) = (\sqrt{3} \cos 2t, -\sqrt{3} \sin 2t, -1)^T$ (5 points).

Find the linearization of this system about $\gamma(t)$, $\dot{\mathbf{x}} = A(t)\mathbf{x}$ (5 points). Show (5 points) that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} e^{-6t} \cos 2t & \sin 2t & 0 \\ -e^{-6t} \sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Find the characteristic exponents and multipliers of $\gamma(t)$ (2 points). What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$ (3 points)?

3. (15 points) Show that the nonlinear system

$$\begin{aligned} \dot{x} &= 3x - y - 4x^3 - 7xy^2, \\ \dot{y} &= x + 3y - 4x^2y - 7y^3, \end{aligned}$$

- (a) (5 points) has at least one periodic orbit in the annulus $\sqrt{3/7} \leq r \leq \sqrt{3}/2$ (for now use the fact, that the only critical point is the origin);
(b) (5 points) there is exactly one orbit inside the annulus;
(c) (5 points) prove that there are no critical points but the origin.

4. (25 points) Solve the initial-value problem

$$tu_t + (x + t^2)u_x = u^3, \quad t > 0, \quad x \in \mathbb{R}, \quad u(t, x)|_{t=2} = x^2 + 2.$$

5. (25 points) Consider the initial value problem (IVP) problem

$$\begin{cases} u_t - \Delta u + \alpha u = f \text{ in } \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} = u_0(\mathbf{x}) \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $\alpha \in \mathbb{R}$ is the constant, $u_0(\mathbf{x})$ and $f(\mathbf{x}, t)$ are in the Schwartz space $S(\mathbb{R}^n)$ for \mathbf{x} (i.e. u_0 and f are infinitely differentiable and all their derivatives in \mathbf{x} decay faster than any power of \mathbf{x} at $|\mathbf{x}| \rightarrow \infty$). Also assume that $f(\mathbf{x}, t)$ is continuous in $t \in [0, \infty)$

(a) (10 points) Show that this IVP is well-posed in L^2 , i.e. $\|u(\cdot, t)\|_{L^2} < \infty$ for any $t \geq 0$.

(b) (15 points) Find the explicit formula for the solution of IVP.