UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Winter 2015

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (20 points) Find a general solution of the following ODE:

$$t^2y'' - t(t+2)y' + (t+2)y = \frac{6-6t+2t^2}{t}e^{2t},$$

given that $y_1(t) = t$ is a solution of the corresponding homogeneous equation.

2. (10 points) For the nonlinear system:

$$\begin{cases} x' &= x - y + x^2 - xy, \\ y' &= -y + x^2. \end{cases} .$$

(a) determine the critical points for the equation;
(b) determine the linearized system for each critical point and discuss whether it can be used to approximate the behaviour of the non-linear system;
(c) what is the type and stability of each critical point?

3. (20 points) Show that the nonlinear system

$$\dot{x} = 3x - y - 4x^3 - 7xy^2,$$
$$\dot{y} = x + 3y - 4x^2y - 7y^3,$$

has at least one periodic orbit in the annulus $\sqrt{3/7} < r < \sqrt{3}/2$ (pay attention, that you have to show that trajectory cannot be on the boundary of the annulus!).

4. (25 points) Solve the following Cauchy problem for a first order PDE:

$$(2x_1 + x_2)u_{x_1} + (x_2 + 1)u_{x_2} = u^2, \qquad u(x_1, x_2)|_{x_2=1} = x_1^2 + 1, \qquad x_1 \geq 0, x_2 \geq 1$$

and find an implicit condition over x_1 and x_2 under which this Cauchy problem has a bounded solution.

5. (15 points) Use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

$$\begin{aligned} u_{tt} - 4u_{xx} = 8,\ x \in (0, l),\ t \in (0, \infty),\ l > 0,\ a = const, \\ u|_{x=0} = u|_{x=l} = 0,\ t \in (0, \infty), \\ u(x, t)|_{t=0} = x(l - x) + \sin\left(\frac{5\pi x}{l}\right), \\ u_t(x, t)|_{t=0} = 0. \end{aligned}$$

Hint: represent solution as $u = v + w$, where v is the solution of nonhomogeneous t-independent problem and w is the solution of t-dependent homogeneous (i.e. with zero right hand side) problem.

6. (10 points) Prove the uniqueness of the solution $u(x, t)$ of the following initial/boundary value problem for the heat equation

$$\begin{aligned} u_t - u_{xx} = 0,\ x \in (0, l),\ t \in (0, \infty),\ l > 0, \\ u_x|_{x=0} = 0,\ u|_{x=l} = 0,\ \ t \in (0, \infty), \\ u(x, t)|_{t=0} = u_0(x). \end{aligned}$$