UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

January 2016

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (15 points) Consider the nonlinear system $\dot{x} = y - y^3$, $\dot{y} = -x - y^2$.

   (a) Find the fixed points and classify them by computing the linearization about each.

   (b) Carefully sketch the phase portrait.

2. (10 points) A gradient system for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a system of the form

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x}) \quad \text{or} \quad \dot{x}_i = -\frac{\partial V}{\partial x_i}, \quad i = 1, \ldots, n$$

   for some smooth function $V : \mathbb{R}^n \to \mathbb{R}$.

   Suppose $\mathbf{x}(t)$ is the solution of a gradient system with initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

   (a) Prove that $V(\mathbf{x}(t)) \leq V(\mathbf{x}_0)$.

   (b) Show that $\dot{x} = -x + 2y - x^3$, $\dot{y} = 2x - y - y^3$ is a gradient system by finding an appropriate function $V(x, y)$.

3. (20 points total) Show that the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= 4x_1 - 8, \\
\dot{x}_2 &= 2x_2 + 2x_3 - 2x_2^3 - \frac{1}{2}x_2 x_3^2, \\
\dot{x}_3 &= -8x_2 + 2x_3 - 2x_2^2 x_3 - \frac{1}{2}x_3^3,
\end{aligned}$$

   has a periodic orbit $\gamma(t) = (2, \sin 4t, 2\cos 4t)^T$ (2 points).
   Find the linearization of this system about $\gamma(t)$, $\dot{\mathbf{x}} = A(t)\mathbf{x}$ (5 points). Show (5 points) that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & \cos 4t & \frac{1}{2}e^{-4t}\sin 4t \\ 0 & -2\sin 4t & e^{-4t}\cos 4t \end{pmatrix}.$$

   Find the characteristic exponents and multipliers of $\gamma(t)$ (5 points). What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$ (3 points)?

4. (20 points) Solve the following Cauchy problems for a first order PDE:

$$(x_2 + x_2^5)v_{x_1} + x_1 v_{x_2} = 0, \qquad v(x_1, x_2)|_{x_2=0} = v_0(x_1).$$

and find a condition for $v_0(x_1)$ for which the problem is well-posed.

5. (20 points total) Consider the initial value problem (IVP) problem

$$\begin{cases} u_t = a u_{xx} + b u_{xxxx} + f \text{ in } \mathbb{R} \times (0, \infty), \\ u|_{t=0} = u_0(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Here a, $b \in \mathbb{R}$ are the constant, the scalar real functions $u_0(x)$ and $f(x,t)$ are in the Schwartz space $S(\mathbb{R})$ for x (i.e. u_0 and f are infinitely differentiable and all their derivatives in x decay faster than any power of x at $|x| \to \infty$). Also assume that $f(x,t)$ is continuous in $t \in [0, \infty)$

(a) (10 points) Find for which values of a, b this IVP is well-posed in L^2 for all times, i.e. $\|u(\cdot, t)\|_{L^2} < \infty$ for any $t \geq 0$.

(b) (10 points) Find the explicit formula for the solution of IVP in integral form which involves $f(x,t)$ and $u_0(x)$.

6. (15 points) Prove the uniqueness of the solution $u(\mathbf{x}, t) \in C^2(\Omega \times [0, \infty))$ of the following initial/boundary value problem

$$\begin{aligned} u_{tt} - \triangle u = 0, \ \mathbf{x} \in \Omega, \ t \in (0, \infty), \\ \mathbf{n} \cdot \nabla u|_{\partial\Omega} = g(\mathbf{x}), \ g \in C^2(\partial\Omega), \\ u(\mathbf{x}, t)|_{t=0} = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, t)|_{t=0} = u_1(\mathbf{x}), \ \mathbf{x} \in \Omega, \ u_0, u_1 \in C^2(\Omega), \end{aligned}$$

where $\Omega \in \mathbb{R}^2$ is the simply connected bounded open set with the smooth boundary $\partial\Omega$ and $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$.
Hint: Use the energy method for the energy functional $e(t) := \int\limits_{\Omega} [u_t^2 + |\nabla u|^2] d\mathbf{x}$.