UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Summer 2016

*Instructions:* There are five (5) problems on this examination. Work all problems for full credit.

1. (15 points) Find the general solution $y = y(t)$ to the ODE

$$t^2 y'' - 3ty' + 4y = \sqrt{t}$$

on the interval $(0, +\infty)$ given that $y_1(t) = t^2$ is a solution of the corresponding homogeneous equation.

2. (25 points) Consider the Hamiltonian function

$$H(x_1, x_2, y_1, y_2) = U(x_1, x_2) + \frac{1}{2}(y_1^2 + y_2^2), \quad U(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1^2 x_2 - \frac{1}{3}x_2^3.$$

   (a) (2 points) Write down corresponding Hamiltonian system.

   (b) (5 points) Find all critical points of the system obtained in (a).

   (c) (6 points) Determine the stability of the critical points in (b) and the types of critical points for the corresponding linearized systems.

   (d) (3 points) Carefully sketch the phase portrait for the system in (a) in the $(x_1, x_2)$ plane.

   (e) (3 points) Find an integral of motion for the system in (a) and demonstrate that it is conserved along the solution trajectories.

   (f) (3 points) Write down a system orthogonal to the one obtained in (a) and explain what it means.

   (g) (3 points) Demonstrate directly the property of orthogonality for the systems in (a) and (f).

3. (20 points) Use the method of separation of variables to find a solution $u = u(x, y)$ to the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the infinite strip $(-\infty, +\infty) \times (-1, 0)$ subject to the boundary conditions

$$\frac{\partial u}{\partial y}(x, 0) = \cos(2x) \quad \text{and} \quad \frac{\partial u}{\partial y}(x, -1) = 0.$$

4. (20 points) Use the method of characteristics to find a solution $u = u(x, y)$ to the PDE

$$u \frac{\partial u}{\partial y} - x \frac{\partial u}{\partial x} = y$$

subject to the Cauchy initial condition $u(1, y) = y + 1$.

5. (20 points) Suppose $f$, $g$, $h$, and $\phi$ are smooth functions. Use the energy method to establish the uniqueness of a solution $u = u(x, t)$ to the PDE

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t)$$

on $(0, 1) \times (0, +\infty)$ subject to the initial condition $u(x, 0) = \phi(x)$ and boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = g(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(1, t) = h(t).$$