UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Winter 2017

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (15 points) Consider the boundary value problem on the interval $0 < x < 3\pi/2$ given by
$$u'' + u = f(x), \quad u(0) = 0, \quad u(3\pi/2) = 0.$$
   (a) (10 points) Find a Green's function and express the solution of the equation in terms of the Green's function.
   (b) (5 points) Solve the equation with $f(x) = x$ using both the Green's function and any other method (*e.g.* variation of parameters, undetermined coefficients, ...).

2. (15 points) Consider the Hamiltonian function (x being the canonical coordinate, y being the canonical momentum)
$$H(x,y) = \frac{1}{2}\,y^2 + U(x), \quad U(x) = \frac{1}{4}\,x^4 + \frac{1}{6}\,x^6.$$
   (a) (3 points) Write down corresponding Hamiltonian system of ODEs.
   (b) (3 points) Check that the origin is an equilibrium point and investigate its stability using a Lyapunov function.
   (c) (3 points) Find an integral of motion (a scalar quantity which doesn't change along trajectories) for the system from (a) and demonstrate that it is conserved along the solution trajectories.
   (d) (3 points) Write down a system orthogonal to the one obtained in (a) and explain what orthogonality means in this case.
   (e) (3 points) Demonstrate directly that the systems in (d) and (a) are orthogonal.

3. (20 points) Consider the nonlinear system
$$\dot{x} = 5\,x - y - 4\,x^3 - 7\,xy^2,$$
$$\dot{y} = x + 5\,y - 4\,x^2 y - 7\,y^3.$$
   (a) (8 points) Show that this system has at least one periodic orbit in the annulus $\sqrt{5/7} \le r \le \sqrt{5}/2$. For now, use the fact that the only critical point is the origin.
   (b) (6 points) Show that there is exactly one orbit inside the annulus.
   (c) (6 points) Prove that there are no critical points except the origin.

4. (20 points) Assuming that $k(5\pi/L)^2 \neq 1$, solve the nonhomogeneous problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + \mathrm{e}^{-t}\sin\left(\frac{5\pi x}{L}\right) + 1, \quad 0 < x < L$$

subject to

$$u(0,t) = t, \quad u(L,t) = t, \quad u(x,0) = \sin\left(\frac{\pi x}{L}\right).$$

You may suppose that all functions are smooth enough.

5. (15 points) Find a solution $u(x,y)$ to the partial differential equation

$$u\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = x$$

subject to $u(x,1) = 2x$. For what values of y is this solution valid?

6. (15 points) Suppose f is a smooth function with compact support and consider the partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0$$

on $-\infty < x < \infty$ and $t > 0$ subject to $u(x,0) = f(x)$. Find a function $g(x,y,t)$ so that a solution to this initial value problem is given by

$$u(x,t) = \int_{-\infty}^{\infty} f(y)g(x,y,t)\,dy.$$

You may find it useful that $\int_{-\infty}^{\infty} e^{-y^2} e^{isy}\,dy = \sqrt{\pi}\,e^{-s^2/4}$.