UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Summer 2017

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (15 points) Locate all critical points of the following system of ODEs

$$\begin{cases} x' &= x^2 - y^2 - 4, \\ y' &= 4y. \end{cases}$$

(a) (5 points) Sketch the phase plane diagram.
(b) (5 points) Discuss the stability properties of all equilibrium points using linearized system.
(c) (5 points) For a critical point with $x > 0$ (if any), find an appropriate Lyapunov function to investigate stability of this system of ODEs in the vicinity of such a point (if exists).
Hint: change to a new coordinate system which moves origin to the critical point may be useful.

2. (15 points) Considering Hamiltonian function (x is canonical coordinate and y is canonical momentum):

$$H(x, y) = \frac{1}{2}y^2 + U(x), \quad U(x) = \frac{1}{6}x^6 + \frac{1}{8}x^8.$$

(a) (3 points) Write down corresponding Hamiltonian system.
(b) (3 points) Check that the origin is an equilibrium point and investigate stability of it using Lyapunov function.
(c) (3 points) Identify an integral of motion (scalar quantity which does not change along the solution trajectories) for system from (a) and demonstrate by direct calculation that it is conserved along the trajectories.
(d) (3 points) Write down system orthogonal to the one obtained in (a), explain what orthogonality means in this case.
(e) (3 points) Demonstrate directly the property of orthogonality of the systems in (d) and (a).

3. (15 points) Consider Sturm-Liouville boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0, \ \ \phi(x)|_{x=0} = \phi(x)|_{x=1} = 0,$$

where $p(x) \neq 0$, $q(x)$ and $\sigma(x) > 0$ are real-valued continuously differentiable functions. This boundary value problem has solution for a discrete set of eigenvalues $\lambda = \lambda_n$, $n =$

$1, 2, \ldots$ such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots < \infty$. For the case of large eigenvalues $\lambda \gg 1$, consider the approximation of the eigenfunction in the form $\phi(x) = A(x)\sin[\psi(x)]$, where $A(x)$ is a slowly changing function in a sense that $|A(x+\Delta x) - A(x)| \ll |A(x)|$ for such Δx that $|\psi(x+\Delta x) - \psi(x)| = O(1)$, where $|\Delta x| \ll 1$. Find approximation for phase $\psi(x)$ and λ in that limit $\lambda \gg 1$.

Hint: you can use a property that eigenfunction corresponding to n-th eigenvalue has exactly $(n-1)$ zeros. A Taylor series expansion of $\psi(x)$ at $x \in (0,1)$ may be helpful.

4. (20 points)Assuming $\gamma(3\pi/L)^2 \neq 1$, use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = \mathrm{e}^{-t} \sin\left(\frac{3\pi x}{L}\right) + x/L + 1, \quad x \in (0, L),$$
$$u|_{x=0} = t, \quad u|_{x=L} = 2t, \quad t \in (0, \infty),$$
$$u(x,t)|_{t=0} = \sin\left(\frac{\pi x}{L}\right),$$

where $\gamma > 0$ and $L > 0$ are real constants.

Hint: represent solution as $u = v + w$, where v is the solution of the initial/boundary-value problem with zero boundary conditions while w takes care of nonzero boundary conditions of u.

5. (15 points) Solve the following Cauchy problem for a first order PDE:

$$\frac{\partial u}{\partial t} - \sin(t)\frac{\partial u}{\partial x} = 3ut^2, \quad u(x,t)|_{t=0} = \tan(x).$$

6. (20 points total) Consider the initial value problem (IVP) problem

$$\begin{cases} u_{tt} = a u_{xx} + b u_t \text{ in } \mathbb{R} \times (0, \infty), \\ u|_{t=0} = u_0(x) \text{ and } u_t|_{t=0} = u_1(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Here a, $b \in \mathbb{R}$ are the constant, the scalar real-valued functions $u_0(x)$ and $u_1(x)$ are in the Schwartz space $S(\mathbb{R})$ for x (i.e. u_0 and $u_1$are infinitely differentiable and all their derivatives in x decay faster than any power of x at $|x| \to \infty$).

(a) (10 points) Find for which values of a, b this IVP is well-posed in L^2 for all times, i.e. $\|u(\cdot, t)\|_{L^2} < \infty$ for any $\infty > t \geq 0$.

(b) (10 points) Find the explicit formula for the solution of IVP in integral form which involves $u_0(x)$ and $u_1(x)$.