UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

August 2019

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (20 points) Locate the equilibrium points of the following ODE system

$$\left\{ \begin{array}{rcl} x' & = & x(x^2+y^2-1) \\ y' & = & y(x^2+y^2-1) \end{array} \right. .$$

Sketch the phase plane diagram, and discuss stability properties of all equilibrium points.

2. (20 points) Show that the nonlinear system

$$\dot{x} = 4x - y - x(4x^2+7y^2),$$
$$\dot{y} = x + 4y - y(4x^2+7y^2),$$

(a) has at least one periodic orbit in the annulus $\sqrt{4/7} \leq r \leq 1$ (for now use the fact, that the only critical point is the origin);
(b) there is exactly one orbit inside the annulus;
(c) prove that there are no critical points but the origin.

3. (10 points) Consider Sturm-Liouville boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0, \ \phi(x)|_{x=0} = \phi(x)|_{x=1} = 0,$$

where $p(x) \neq 0$, $q(x)$ and $\sigma(x) > 0$ are real-valued continuously differentiable functions. This boundary value problem has solution for a discrete set of eigenvalues $\lambda = \lambda_n$, $n = 1, 2, \ldots$ such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots < \infty$. For the case of large eigenvalues $\lambda \gg 1$, consider the approximation of the eigenfunction in the form $\phi(x) = A(x)\sin[\psi(x)]$, where $A(x)$ is a slowly changing function in a sense that $|A(x+\Delta x) - A(x)| \ll |A(x)|$ for such Δx that $|\psi(x+\Delta x) - \psi(x)| = O(1)$, where $|\Delta x| \ll 1$. Find approximation for phase $\psi(x)$ and λ in that limit $\lambda \gg 1$.

Hint: you can use a property that eigenfunction corresponding to n-th eigenvalue has exactly $(n-1)$ zeros. A Taylor series expansion of $\psi(x)$ at $x \in (0, 1)$ may be helpful.

4. (15 points) Consider a heat equation in a right circular hollow cylinder (cylindrical shell) of the radius a and the height H. The thickness of the curved shell of the cylinder is assumed to be much smaller than a, i.e. its thickness is much less than the radius and can be neglected. The shell in the cylindrical coordinates is described by the radius $r = a$, the vertical height y and the azimuthal angle θ around y axis. The lower side (bottom) of the cylinder has the coordinate $y = 0$, and the upper side (top) has the coordinate $y = H$. The distribution of temperature in the shell is given by $u(\theta, y, t)$. The curved shell is assumed to have the perfect heat insulation except the edges. The upper edge of the shell is assumed to be cooling in the atmosphere of temperature 0 following the Newton's law of cooling $u(\theta, H, t) - \frac{\partial u}{\partial y}(\theta, y, t)|_{y=H} = 0$.
(a) (10 points) Find the equilibrium solution of the heat equation:
$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad r = a, \quad \theta \in [0, 2\pi), \ \ y \in [0, H],$$
if the distribution of temperature at lower edge does not depend on time $u(\theta, 0, t) = f(\theta)$, where $f(\theta)$ is the given function which is assumed to be smooth enough. **Hint**: one can unroll the shell to the rectangular by cutting it vertically together with the using of the appropriate boundary condition along the cut.
(b) (5 points) What will be the solution of the problem (a) if $f(\theta) = T = const$?

5. (15 points) Solve the following Cauchy problem for a first order PDE:
$$(xy + 1)\,[xu_x - yu_y] = (x^2 + y^2)u^2, \qquad u(x, y)|_{y=1} = x - 2.$$

6. (20 points total) Consider the initial value problem (IVP) problem
$$\begin{cases} u_t = au_{xx} + bu_{xxxxxx} + f \text{ in } \mathbb{R} \times (0, \infty), \\ u|_{t=0} = u_0(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$
Here a, $b \in \mathbb{R}$ are the constant, the scalar real functions $u_0(x)$ and $f(x, t)$ are in the Schwartz space $S(\mathbb{R})$ for x (i.e. u_0 and f are infinitely differentiable and all their derivatives in x decay faster than any power of x at $|x| \to \infty$). Also assume that $f(x, t)$ is continuous in $t \in [0, \infty)$

(a) (10 points) Find for which values of a, b this IVP is well-posed in L^2 for all times, i.e. $\|u(\cdot, t)\|_{L^2} < \infty$ for any $t \geq 0$.

(b) (10 points) Find the explicit formula for the solution of IVP in integral form which involves $f(x, t)$ and $u_0(x)$.