Mathematics and Statistics, Unversity of New Mexico

MS/PhD Qualifying Examination, Fall 2020

Ordinary and Partial Differential Equations

Of the following six problems, please do five of your choice. Please start each problem on a new page labeled with the problem number and your secret ID code. Please motivate your answers and show your work.

1. (20 points) Consider the initial value problem (IVP) $\dot{y} = -y^2$, $y(0) = \eta \in \mathbb{R}$.

(a) (5 points) Sketch the direction field, and discuss the stability of the equilibrium solution $y = 0$.

(b) (5 points) Is the solution to this IVP unique? Why?

(c) (5 points) Denote the solution $y(t)$ of the IVP by $y(t) = \Phi(t, \eta)$. Find the maximal interval (α, β) of existence for $\Phi(t, \eta)$ as a function of η. Note: $\alpha = -\infty$ and $\beta = \infty$ are possible values.

(d) (5 points) Show that $\Phi(t_0 + t, \eta) = \Phi(t, \Phi(t_0, \eta))$ whenever they are defined.

2. (20 points) Consider the IVP

$$x'' + x - \epsilon x^3 = 0, \qquad x(0) = A, \qquad x'(0) = 0,$$

where $\epsilon > 0$ is small.

(a) (8 points) Introduce $y = x'$ and consider the associated first-order nonlinear system. Sketch its phase-plane portrait and determine the range of A such that solutions are periodic.

(b) (10 points) Determine the first two terms in the regular perturbation expansion

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots,$$

where x_0 is the zeroth-order solution and ϵx_1 is the first-order correction. The first-order correction ϵx_1 has a *secular term* which is unbounded as $t \to \infty$. Identify the secular term in x_1. **Hint:** Work with the second-order scalar equation, not the first-order system. Substitute the expansion into the equation and ignore all terms quadratic and higher in ϵ. To solve for x_1, you may use the identity

$$\cos^3 t = \tfrac{1}{4}\cos 3t + \tfrac{3}{4}\cos t.$$

(c) (2 points) The previous two items point to an apparent paradox. What is the paradox and what is its resolution? **Hint:** $\sin(\epsilon\theta) \sim \epsilon\theta$ and $\cos(\epsilon\theta) \sim 1$, both to first order in ϵ. Therefore, to the same order $\cos\phi + \epsilon\theta\sin\phi \sim \cos(\phi - \epsilon\theta)$. Use these observations to combine the zeroth-order solution with the secular term.

3. (20 points)

(a) (12 points) Consider the system

$$q' = e^p - 1, \qquad p' = 1 - e^q.$$

Show that the system is Hamiltonian and choose the Hamiltonian function such that $H(0, 0) = 0$. Argue that the solutions are periodic orbits surrounding the fixed point $(0, 0)$. **Hint:** You may assume that the level sets $H(p, q) = C > 0$ are closed curves.

(b) (8 points) Consider the system

$$q' = e^p - 1 - e^q, \qquad p' = 1 - e^q - e^p.$$

Use Bendixson's criterion to show that the system has no periodic orbits. For full credit explain why Bendixson's criterion is sufficient.

4. (20 points) Use the method of separation of variables and eigenfunction expansion to solve the following inhomogeneous initial-boundary value problem for the heat equation.

$$\begin{aligned} u_t - u_{xx} &= 0, \qquad x \in (0,1), \quad t > 0 \\ u(x,0) &= x(1-x), \qquad u(0,t) = t = u(1,t) \end{aligned}$$

5. (20 points) Consider the following initial value problem:

$$\begin{aligned} u_t &= u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= e^{-x^2}, \qquad x \in \mathbb{R}. \end{aligned}$$

(a) (15 points) Use Fourier transformation in x to find the solution $u = u(x,t)$ to the problem. For full credit evaluate encountered integrals.

(b) (5 points) Use Laplace transformation in t to find the corresponding resolvent equation (i.e. an ODE in x).

6. (20 points) Consider the following initial-boundary value problem for the wave equation.

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= f(x,t), && \text{for } (x,t) \in (0,1) \times (0,T] \\ u(x,0) = \phi_1(x), \quad u_t(x,0) &= \phi_2(x), && \text{for } x \in [0,1] \\ u_x(0,t) = g(t), \quad u_x(1,t) &= h(t), && \text{for } t \geq 0, \end{aligned}$$

where $T > 0$ is a finite terminal time. Suppose that all data are smooth and compatible. Use the energy method to prove that there exists at most one solution $u = u(x,t)$ to the problem.