Mathematics and Statistics, Unversity of New Mexico

MS/PhD Qualifying Examination, Fall 2021

Ordinary and Partial Differential Equations

Of the following six problems, please do five of your choice. Please start each problem on a new page labeled with the problem number and your secret ID code. Please motivate your answers and show your work.

1. (20 points) Consider the system

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}, \qquad A(t) = \begin{pmatrix} -2\sin t\cos t & \cos^2 t - \sin^2 t - 1 \\ \cos^2 t - \sin^2 t + 1 & 2\sin t\cos t \end{pmatrix},$$

where the coefficient matrix obeys $A(t+\pi) = A(t)$.

(a) (10 points) The following is a *fundamental solution matrix* for the system:

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & -e^{-t}(\cos t - \sin t) \end{pmatrix}.$$

State what this means, *but no need for a tedious verification.* Use $\Phi(t)$ to construct the *principal solution matrix* (or *state transition matrix*) $\Pi(t, t_0)$ for initial data specified at initial time $t_0 = 0$, and show that it can be written in the following real *Floquet normal form.*

$$\Pi(t,0) = P(t)e^{Qt} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \exp\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t\right)$$

Regarding (the real version of) the *Floquet Theorem*, what are the key observations pertaining to $P(t)$? **Hint:** Work out the matrix exponential to compare the given expression for $\Pi(t,0)$ with the one that follows from $\Phi(t)$.

(b) (5 points) Write down the monodromy matrix $M(t_0)$ based at $t_0 = 0$. Next, find the Floquet multipliers $\mu_i, i = 1,2$ for the system, so that there exists solutions of the form

$$\mathbf{y}_i(t+\pi) = \mu_i \mathbf{y}_i(t), \qquad i = 1,2.$$

(c) (5 points) Does the system have solutions of period π? It is sufficient to consider the monodromy matrix $M(t_0)$ with $t_0 = 0$. Why?

2. (20 points)

(a) (5 points) State the *Poincaré-Bendixson Theorem.*

(b) (15 points) The planar system

$$\frac{dx}{dt} = x - y - x^3, \qquad \frac{dy}{dt} = x + y - y^3$$

has only one critical point at the origin; you may assume this. Show that this system has a periodic orbit in the annular region $A = \{(x,y) : 1 < \sqrt{x^2+y^2} < \sqrt{2}\}$.

Hint: Convert to polar coordinates. Then show for small $\epsilon > 0$ that $\dot{r} < 0$ on the circle $r = \sqrt{2} + \epsilon$, and $\dot{r} > 0$ on the circle $r = 1 - \epsilon$. Argue that this means there is a limit cycle in the closure $\bar{A} = \{(x,y) : 1 \leqslant \sqrt{x^2+y^2} \leqslant \sqrt{2}\}$. Next, argue that no limit cycle can have a point in common with either of the circles $r = 1$ or $r = \sqrt{2}$. Note also that $\sin(2u) = 2\sin u\cos u$ and $\cos(2u) = \cos^2 u - \sin^2 u$.

3. (20 points) Given that $y_1(t) = e^t$ is a solution of the corresponding homogeneous equation, find the general solution of the following ODE:

$$t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y = t^2.$$

4. (20 points)

(a) (3 points) What is a harmonic function?

(b) (9 points) Precisely state the following properties of harmonic functions.

- mean-value property
- weak maximum principle
- strong maximum principle

(c) (8 points) Consider the following function:

$$u(x_1, x_2) = \ln(x_1^2 + x_2^2), \qquad (x_1, x_2) \in \bar{U} \subset \mathbb{R}^2,$$

where $\bar{U}$ is the closed annulus $\{(x_1, x_2) : \frac{1}{2} \leqslant \sqrt{x_1^2 + x_2^2} \leqslant 1\}$.

- Verify that u is harmonic.
- Verify that both weak maximum principle and weak minimum principle hold for u. Justify your answers.

5. (20 points) Consider the following initial-boundary value problem for the wave equation.

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= f(x,t), && \text{for } (x,t) \in (0,1) \times (0,T] \\ u(x,0) = \phi_1(x), \quad u_t(x,0) &= \phi_2(x), && \text{for } x \in [0,1] \\ u_x(0,t) = g(t), \quad u_x(1,t) &= h(t), && \text{for } t \geqslant 0, \end{aligned}$$

where $T > 0$ is a finite terminal time. Suppose that all data are smooth and compatible. Use the energy method to prove that there exists at most one solution $u = u(x,t)$ to the problem.

6. (20 points) Solve the following problem using characteristics:

$$\begin{aligned} u\,u_{x_1} + u_{x_2} &= 1, && x_1 \in \mathbb{R},\ x_2 > 0 \\ u &= \frac{1}{2}x_1, && x_1 \in \mathbb{R},\ x_2 = 0 \end{aligned}$$