Of the following six problems, please do five of your choice (100 full points). Do not attempt all six problems; if you do so, only the first five will be graded. It is recommended that you mark an X through the problem you do not want graded.

Please start each problem on a new page labeled with the problem number. Write your secret code on each page and number all pages in order.

Motivatite and justify your answers clearly and show your work.

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1. (20 points) Consider the nonlinear ODE system

$$\dot{x} = -2x + x^2 - y^2,$$
$$\dot{y} = -2y.$$

(a) (10 points) Find all critical points and analyze the stability of linearized systems around each of these points.
(b) (5 points) Carefully sketch the phase plane portrait.
(c) (5 points) Analyze the nonlinear stability of the origin by finding the Liapunov function. Hint: Look for the Liapunov function in the form of the quadratic polynomial in x and y.

2. (20 points) Consider the nonlinear ODE system

$$\dot{x} = y + 4x - 7x^3 - 4xy^2,$$
$$\dot{y} = 4y - x - 7x^2y - 4y^3.$$

(a) (10 points) Show that the nonlinear system has at least one periodic orbit in the annulus $\sqrt{4/7} \le r \le 1$ (for now use the fact, that the only critical point is the origin).
(b) (5 points) Show that there is exactly one periodic orbit inside the annulus.
(c) (5 points) Prove that there are no critical points but the origin.

Hint: use polar coordinates.

3. (20 points) Consider the nonlinear ODE system

$$\dot{x}_1 = 4x_1 - 8,$$
$$\dot{x}_2 = 2x_2 + 2x_3 - 2x_2^3 - \frac{1}{2}x_2 x_3^2,$$
$$\dot{x}_3 = -8x_2 + 2x_3 - 2x_2^2 x_3 - \frac{1}{2}x_3^3.$$

(a) (2 points) Show that the nonlinear system has a periodic orbit $\gamma(t) = (2, \sin 4t, 2\cos 4t)^T$.
(b) (5 points) Find the linearization of this system about $\gamma(t)$, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, $\mathbf{x} = (x_1, x_2, x_3)^T$.

(c) (5 points) Show that the fundamental matrix $\Phi(t)$, $\Phi(0) = I$, for this linear system is given by

$$\Phi(t) = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & \cos 4t & \frac{1}{2}e^{-4t}\sin 4t \\ 0 & -2\sin 4t & e^{-4t}\cos 4t \end{pmatrix}.$$

What is the period of that linearized system?

(d) (5 points) Find the characteristic exponents and multipliers of $\gamma(t)$.

(e) (3 points) What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

4. (**20 points**) Let u be a smooth function of $(\mathbf{x},t)$ in $\mathbb{R}^n \times (0,\infty)$ that solves the heat equation $u_t = \Delta u$.

   (a) (9 points) Show that $u_\lambda := u(\lambda\,\mathbf{x}, \lambda^2\,t)$ also solves the heat equation for any $\lambda \in \mathbb{R}$.

   (b) (11 points) Show that $v(\mathbf{x}, t) := \mathbf{x} \cdot Du(\mathbf{x}, t) + 2\,t\,u_t(\mathbf{x}, t)$ also solves the heat equation. Hint: use part (a).

5. (**20 points**) Consider the homogeneous wave equation in 1D:

$$u_{tt} - u_{xx} = 0, \qquad x \in \mathbb{R}, \qquad t > 0,$$

with the initial conditions

$$u(x, 0) = g(x), \qquad u_t(x, 0) = h(x), \qquad x \in \mathbb{R}.$$

   (a) (9 points) Let $\alpha = x + t$ and $\beta = x - t$. Show that the wave equation $u_{tt} - u_{xx} = 0$ holds if and only if $u_{\alpha\beta} = 0$, that is, the mixed derivative of u with respect to α and β vanishes.

   Hint: First write (x, t) in terms of (α, β).

   (b) (11 points) Use the equivalent eqution $u_{\alpha\beta} = 0$ to derive d'Alembert's formula.

   Hint: First show that the solution of $u_{\alpha\beta} = 0$ takes the form $u = F(\alpha) + G(\beta)$, and then find F and G in terms of the initial data.

6. (**20 points**) Solve the following problem using characteristics:

$$u\,u_{x_1} + u_{x_2} = 1, \qquad x_1 \in \mathbb{R},\ x_2 > 0$$

$$u = \frac{1}{2}x_1, \qquad x_1 \in \mathbb{R},\ x_2 = 0$$