Of the following six problems, please do five of your choice (100 total points). Please do not include solutions to all six problems; if you do so, then only the first five will be graded. It is recommended that you mark an X through the problem you do not want graded.

Please start each problem on a new page labeled with the problem number. Please write your secret code on each page and number all pages in order. Please justify your answers and show your work.

1. (20 points) Assume for $A \in \mathbb{R}^{n \times n}$ that $\operatorname{Re} \lambda \neq 0$ for any eigenvalue λ of A. Let $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ be continuous and T-periodic: $\mathbf{g}(t+T) = \mathbf{g}(t)$. Show that the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t)$$

has a unique solution which is *T*-periodic. Hint: for integer $\ell > 0$ consider the Duhamel principle for initial data at t_0 and $t_0 + \ell T$.

2. (20 points) Consider the system

$$\frac{dx}{dt} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}, \qquad \frac{dy}{dt} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

Show that the fixed point (1,0) is not Liapunov stable, even though it satisfies

$$\lim_{t\to\infty} \|\Phi(t,\mathbf{x}) - \mathbf{x}_0\| = 0 \text{ for all } \mathbf{x} \in U(\mathbf{x}_0)$$

for any small open neighborhood $U(\mathbf{x}_0)$ of the fixed point \mathbf{x}_0 . Also discuss the time taken to reach the equilibrium point. Hint: examine the system in polar coordinates and note that $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$.

3. (20 points) Consider the nonlinear system

$$\frac{dx_1}{dt} = -x_1, \qquad \frac{dx_2}{dt} = x_2 - x_1^2.$$

(a) Find the solution to this equation given the generic initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

(b) Write down the linearization $d\mathbf{y}/dt = A\mathbf{y}$ of this system about the fixed point, and find its general solution

(c) The origin in this example is a "hyperbolic equilibrium" (define what this means); therefore, the Hartman Grobman Theorem applies. Using the principle solution matrix e^{At} for the linearized system, the flow $\phi_t(\mathbf{x})$ for the nonlinear system, the initial data \mathbf{x}_0 , and a diffeomorphism H defined on a neighborhood of the origin, write down the key conclusion of the Hartman Grobman Theorem, and find the particular diffeomorphism $\mathbf{y} = H(\mathbf{x})$ for the example above.

4. (20 points) Let $U \subset \mathbb{R}^d$ be a connected open, bounded domain. Assume that $u \in C^2(U) \cap C(\overline{U})$ satisfies the following boundary-value problem:

$$\Delta u = 0, \quad \text{in } U$$
$$u = g, \quad \text{on } \partial U$$

where $g \ge 0$. Prove that u is strictly positive everywhere in U if g is non-zero somewhere on ∂U .

5. (20 points) Solve the following problem using characteristics:

$$u - u_{x_1} u_{x_2} = 0, \qquad x_1 \in \mathbb{R}, \quad x_2 > 0$$
$$u = x_1^2, \qquad x_1 \in \mathbb{R}, \quad x_2 = 0$$

6. (20 points) Consider the solution u to the Cauchy problem for the homogenous wave equation in n spatial dimensions:

 $u_{tt} - c\Delta = 0$ on $\mathbb{R}^n \times (0, \infty)$ subject to $u(\cdot, 0) = g, u_t(\cdot, 0) = h$ on \mathbb{R}^n ,

for sufficiently smooth initial data g and h.

(a) For n = 1 derive the d'Alembert formula

$$u(x,t) = \frac{1}{2} \left[g(x+ct) + g(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(\xi) d\xi$$

for the solution of problem.

(b) The analogous representation for n = 3 is the *spherical means formula* (a special case of the Kirchhoff integral):

$$u(\mathbf{x},t) = t \int_{\partial B_{\mathbf{x}}(ct)} h dS + \frac{\partial}{\partial t} \left(t \int_{\partial B_{\mathbf{x}}(ct)} g dS \right).$$

Here, $\partial B_{\mathbf{x}}(ct)$ is the set $\{\mathbf{y} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{y}\| = ct\}$, and f denotes proper average. If dS is the measure for a sphere of radius r, then f includes the reciprocal of $4\pi r^2$. Using the d'Alembert and spherical means formulas, discuss the relationship between regularity of the initial data and the solution for the n = 1, 3 Cauchy problems. Hint: parameterize the n = 3 representation using $\mathbf{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ and ϕ are the polar and azimuthal angles.

(c) Qualitatively discuss the domain of dependence for solutions of the 1, 2, and 3-space dimensional wave equations. How is this related to the finite speed of propagation?