Qualifying Exam

ODE/PDE

August 2023

Secret ID Code ________________________________

Directions

- This exam has six problems. Complete any **five** problems of your choosing.
- Start each problem on a new page, labeled with the problem number and your secret ID code.
- Motivate your answers and show your work.

Point totals

Problem	1	2	3	4	5	6	total
Score	/20	/20	/20	/20	/20	/20	/100

1. (20 Points) Consider the system

$$x' = 2x - x^2 - xy,$$

$$y' = x - y.$$

Find and classify all fixed points and sketch the phase portrait.

2. (20 Points) Consider the nonlinear ODE

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -xy - y \\ x + x^2 \end{bmatrix},$$

on the domain $E \subset \mathbb{R}^2 = \{(x,y)|x > -1\}$.

(a) Perform linear stability analysis of the fixed point at the origin. What conclusions can you draw about the stability of this fixed point?

(b) Prove that solution trajectories in a neighborhood around the origin are closed circles.

(c) Is the origin stable, asymptotically stable, or unstable?

3. (20 Points) Consider the non-linear ODE

$$x' = xy,$$

$$y' = -y - x^2.$$

(a) Find the stable and center subspaces of the linearized system at the origin.

(b) Use the Center Manifold Theorem to find a 3rd order Taylor Polynomial approximation of the center manifold.

(c) Use the Center Manifold Theorem to classify the fixed point at the origin, then sketch the phase portrait in a neighborhood around it.

4. (20 Points) Prove the uniqueness of solution of the Poisson's equation

$$\nabla^2 \phi = f(\mathbf{r}),$$

in the compact domain $\mathbf{r} \in \Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ subject to Neumann boundary condition $\frac{d\phi}{d\mathbf{n}}|_{\partial\Omega} = \psi_N(\mathbf{r})$ (where $\frac{d}{d\mathbf{n}}$ is a derivative along a normal to the boundary), if the value of the solution is specified in one point inside of the domain. Here $\psi_N \in C^1(\partial\Omega)$ and $\phi \in C^2(\Omega)$.

5. (20 Points) Find the vertical displacement $u(r, \theta, t)$ of the vibrating membrane in the form of the quarter of the unit disc attached at its boundary to the horizontal plane and parametrized by the polar coordinates (r, θ). The dynamics of the membrane is described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

Fixed boundaries correspond to

$$u(r, 0, t) = 0, \; u(r, \pi/2, t) = 0, \; 0 \le r \le 1,$$

$$u(1, \theta, t) = 0, \; 0 \le \theta \le \pi/2.$$

Initial conditions are

$$u(r, \theta, 0) = \alpha(r, \theta), \; \frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta).$$

You need to do full derivation, including proof of positiveness of eigenvalues.

6. (20 Points) Assuming $\gamma(3\pi/L)^2 \neq 1$, use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = \mathrm{e}^{-t} \sin\left(\frac{3\pi x}{L}\right) + x/L + 1, \quad x \in (0, L),$$
$$u|_{x=0} = t, \quad u|_{x=L} = 2t, \quad t \in (0, \infty),$$
$$u(x,t)|_{t=0} = \sin\left(\frac{\pi x}{L}\right),$$

where $\gamma > 0$ and $L > 0$ are real constants.

Hint: represent solution as $u = v + w$, where v is the solution of the initial/boundary-value problem with zero boundary conditions while w takes care of nonzero boundary conditions of u.