January 10, 2007

Geometry/Topology Qualifying Exam
Department of Mathematics & Statistics
University of New Mexico

Instructions: Do 7 of the following 8 problems. Show all your work.

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(1) Let X, Y be compact. Show that $X \times Y$ is compact.

(2) Let X, Y be simply connected. Show that $X \times Y$ is simply connected.

(3) Let $\sim$ be an equivalence relation on a topological space S such that the natural projection $\pi : S \to S/\sim$ is an open map. Define the set $E = \{(x, y) \in S \times S \mid x \sim y\}$. Show that if E is closed in $S \times S$, then the quotient space $S/\sim$ is Hausdorff.

(4) Consider the (infinite) Möbius band M defined as the identification space of $[0, 1] \times \mathbb{R}$ with $(0, y)$ identified with $(1, -y)$, that is, $M = ([0, 1] \times \mathbb{R})/\sim$. Prove that the projection map $\pi : M \to S^1$ defined by $\pi([t, y]) = t$ gives M the structure of a rank one vector bundle over the circle S^1. (Here $[t, y]$ denotes the equivalence class of $(t, y) \in [0, 1] \times \mathbb{R}$.)

(5) Show that the 3-sphere S^3 is parallelizable, that is has a global frame.

(6) Let $f : \mathbb{R}^3 - \{(0, 0, z)\} \to \mathbb{R}$ be given by

$$f(x, y, z) = (2 - \sqrt{x^2 + y^2})^2 + z^2.$$

Show that 1 is a regular value of f and identify $f^{-1}(1)$.

(7) Prove that every vector field on a compact manifold is complete.

(8) Determine the integral curves of the vector field $X = x\partial_y - y\partial_x$ on $\mathbb{R}^2$.