August 2007

Geometry/Topology Qualifying Exam
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Instructions: Do the following 7 problems. Show all your work. Notice the indicated relative value of each problem.

(1) (10 points) Find an open covering of the set $S = \{1/n \,:\, n = 1, 2, \ldots\} \subset \mathbb{R}$ that does not contain a finite subcovering. Is it possible to do the same for the set $S \cup \{0\}$?

(2) (20 points) On the unit sphere $\mathbb{S}^n$, define the relation that identifies a point x with its antipodal $-x$. Let $\mathbb{RP}^n$ be the quotient space. Prove that $\mathbb{RP}^n$ is a Hausdorff space.

(3) (10 points) Compute the fundamental group of the doubly punctured sphere $\mathbb{S}^2 \setminus \{p, q\}$. Here p and q are any two given points on $\mathbb{S}^2$.

(4) (20 points) Let X be a path-connected topological space such that $\pi_1(X, x_0)$ is a finite group. Prove that any continuous function

$$f : X \to \mathbb{S}^1 \times \mathbb{S}^1$$

is homotopic to a constant map.

(5) (20 points) Let G be a topological group, X a Hausdorff space, and $G \times X \to X$ a continuous action of G on X. Let $I(x) = \{g \in G \,:\, gx = x\}$ and $O(x) = \{y \in X \,:\, y = gx \text{ for some } g \in G\}$.
 (a) Show that $I(x)$ is a closed subgroup of G.
 (b) Show that if $y \in O(x)$, then $I(x)$ and $I(y)$ are conjugate subgroups of G.
 (c) Show that if G is compact, then $G/I(x)$ is homeomorphic to $O(x)$.

(6) (20 points) Let $\mathbb{GL}(n)$ denote the Lie group of non-singular $n \times n$ matrices, and let

$$\mathbb{SL}(n) = \{A \in \mathbb{GL}(n) \,:\, \det A = 1\}\,.$$

 (a) Show that $\mathbb{SL}(n)$ is a Lie group.
 (b) Show that $\mathbb{SL}(n)$ is connected and noncompact.
 Hint (for connectedness): use elementary matrices to show that $\mathbb{SL}(n)$ is path-connected.

(7) (20 points) Let $f : M^{(n)} \to N^{(k)}$ be a smooth map between differentiable manifolds. Suppose that

$$df(x) : T_xM \to T_pN$$

is surjective for all $x \in f^{-1}(p)$. Show that $f^{-1}(p)$ is an $n-k$ dimensional submanifold of M.

Hint: use the fact that if $g : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth map such that $g(0) = 0$ and $Dg(0)$ has rank k, then there exists a local diffeomorphism $\varphi : U \ni 0 \to \varphi(U) \subset \mathbb{R}$ such that $g \circ \varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.