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Geometry/Topology Qualifying Exam
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Instructions: Do the following 8 problems. Show all your work.

(1) Recall that a metric space X is *second countable* if it has a countable basis of open sets. Prove that a metric space is second countable if and only if X has a countable dense subset.

(2) Let M be a set, and suppose that

$$d : M \times M \to \mathbb{R}$$

is a function such that

a) $d(p, q) = 0$ if, and only if, $p = q$, and

b) $d(p, r) \leq d(r, q) + d(q, p)$ for any triple $p, q, r \in M$.

Is it true then that the collection of sets

$$\{B_d(p, \varepsilon) = \{q : \ d(p, q) < \varepsilon\}\}_{\varepsilon > 0, q \in M}$$

defines the basis for a topology on M? If so, show why.

(3) Let $X = \mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}$ is the set of real numbers with the usual topology, and $\mathbb{R}_d$ equals $\mathbb{R}$ as sets, but with the discrete topology.

(a) Prove that X is Hausdorff.

(b) Prove that X is locally Euclidean, that is each point has an open set that is homeomorphic to an open set of $\mathbb{R}^2$.

(c) Is X 2nd countable? Explain.

(4) Let M be a smooth manifold of dimension n and $F : M \to \mathbb{R}^k$ a smooth map with $k < n$. Suppose that $c \in \mathbb{R}^k$ is a regular value, that is, the differential map $F_* : T_pM \to T_{F(p)}\mathbb{R}^k$ is surjective for all $p \in F^{-1}(c)$. Show that the level set $F^{-1}(c)$ is a closed embedded submanifold of M.

(5) Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

(6) Given a space X and a path-connected subspace A containing the base point p, show that the map $\pi_1(A,p) \to \pi_1(X,p)$ induced by the inclusion $A \hookrightarrow X$ is surjective if every path in X with endpoints in A is homotopic to a path in A.

(7) For any connected compact submanifold M of $\mathbb{R}^n$ with its Euclidean space structure, define a metric function by

$$d(p,q) = \inf_{\gamma} L(\gamma)\,,$$

where the infimum is taken among all the differentiable curves in M that begin at p and end at q, and where $L(\gamma)$ stands for the lenght of such a curve. Let M_1 and M_2 be two connected compact one dimensional submanifolds of $\mathbb{R}^n$. Show that there exists an isomorphism

$$f : M_1 \to M_2$$

such that

$$d(p,q) = d(f(p), f(q)) \quad \text{for all } p, q \in M_1$$

if, and only if, the length of M_1 is equal to the length of M_2.

(8) Let G be a Lie group acting smoothly on a smooth manifold M.

(a) Show that the quotient map $\pi : M \to M/G$ is open.

(b) Assume also that G is compact and acts freely on M. Show that π is a smooth submersion.

(c) Under the same assumption as in (b) describe the local coordinate charts that make the quotient M/G a smooth manifold.