January, 2012

Geometry/Topology Qualifying Exam
Department of Mathematics & Statistics
University of New Mexico

Instructions: Do the following problems. Show all your work.

1. Let (X, d) be a metric space and S a set. Show that $\mathcal{B}(S : X)$ is a complete metric space iff X is complete. Here, $\mathcal{B}(S : X)$ is the space of bounded functions from S to X with the metric defined by $\rho(f, g) = \sup_{s \in S} d(f(s), g(s))$.

2. A topological space X is called *locally connected* if for each point $x \in X$ and open neighborhood U of x, there is a connected open neighborhood V of x contained in U. Show that a connected component of a locally connected topological space is open.

3. a) Show that every smooth submersion map $F : M \to N$ between two smooth manifolds M and N is an open map.
   b) Show that a surjective submersion F between two smooth manifolds is a quotient map, that is, show that a set in the target space is open iff the pre-image is open.

4. a) Give a definition of a smooth manifold to be orientable.
   b) Show that the tangent bundle of a smooth manifold is an orientable manifold (prove only the orientability claim).

5. Let G be the group of counter clock-wise rotations around the z-axis (when viewed from above the xy-plane) in the three dimensional Euclidean space with a fixed Cartesian coordinate system.
   a) Give a representation of G as a group of 3 by 3 matrices.
   b) Find the infinitesimal generator of G.

6. Let M and N be smooth manifolds and G a Lie group acting smoothly on both M and N, so that the action on M is transitive.
   a) Show that if $F : M \to N$ is a smooth equivariant map with respect to the action of G, i.e., $F(g \cdot p) = g \cdot F(p)$ for all $p \in M$, then F has constant rank.
   b) Show that the level sets of F are closed embedded submanifolds of M.

7. Let $\theta = xdy - ydx + dz$ be a one-form on $\mathbb{R}^3$.
   a) Show that the kernel of θ defined as the set of all tangent vectors X to $\mathbb{R}^3$ annihilated by θ i.e. $\theta(X) = 0$ is a rank two vector bundle over $\mathbb{R}^3$.
   b) Let H be the vector bundle defined in part a). Is H a trivial vector bundle?
   c) Is H integrable, i.e., is H the tangent space of a submanifold of $\mathbb{R}^3$?

8. Let X be a Hausdorff topological space. Show that points are closed in X.

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