Geometry/Topology Qualifying Exam, August, 2013
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Instructions: Do the following problems. Show all your work.

1) Show that if U is an open and F is a closed subset in a topological space X, then $U \setminus F$ is open.
2) Show that if we consider $\mathbb{R}$ with the co-finite topology, then every sequence $\{x_n\}$ of *different* real numbers is convergent and to every real number. In particular, limits are not unique.
3) Show that if $f : K \to Y$ is an injective continuous map of a compact topological space K to a Hausdorff space Y, then f is a homeomorphism between K and its image $f(K) \subset Y$ endowed with the subspace (relative) topology, i.e., f is a topological embedding.
4) Consider the cyclic group C of order two generated by the identity i and the reflection ρ, $\rho(x) = -x$, on the $n-$dimensional sphere S. Find $\pi_1(S/C)$ for $n > 1$.
5) Let M be a smooth manifold and f_i, $i = 1, \dots, k$, be smooth functions, $f_i \in \mathcal{S}(M)$, such that their differentials are linearly independent, i.e., at every $p \in M$ the linear maps $df_i|_p$. $i = 1, \dots, k$ are linearly independent elements of T_p^*M. Show that $S = \{p \in M \mid f_1(p) = \cdots = f_k(p) = 0\}$ is an embedded submanifold of M. What is the dimension of S?
6) Let S be a circle in the plane. For any connected arc of the unit circle which is not the whole circle let ϕ be an angle coordinate, i..e, measures the angle from some fixed point on the circle .
   a) Show that there is a smooth (global) 1-form $\theta \in \Omega^1 S$ such that $\theta = d\phi$.
   b) Is θ a closed form?
7) Let M be a smooth manifold with boundary ∂M. Show that there exists a smooth transversal to the boundary vector field, i.e., a smooth vector field which is not tangential to the boundary (at the boundary points). Hint: piece together using a partition of unity vector fields transversal to the boundary defined on boundary charts and any other defined on interior charts.
8) Let $\mathcal{D} \subset TM$ be a sub-bundle of the tangent bundle such that there is a nowhere vanishing 1-form $\theta \in \Omega^1 M$ for which for every $p \in M$ the fiber of $\mathcal{D}$ over p is $\ker \theta|_p \equiv \mathcal{D}_p \subset T_pM$. Show that $\mathcal{D}$ is integrable, i.e., $X, Y \in \Gamma(\mathcal{D})$ implies $[X, Y] \in \Gamma(\mathcal{D})$ if and only if $d\theta(X, Y) = 0$ for every $X, Y \in \Gamma(\mathcal{D})$. Note: in other words, $\mathcal{D}$ is integrable iff the ideal generated by θ in the exterior algebra of M is a differential ideal.
9) Show that the definition $[\omega_r] \wedge [\omega_k] \overset{def}{=} [\omega_r \wedge \omega_k]$ is well defined and turns $H^*_{dR}(M)$ into a ring. Here, $\omega_l \in \Omega^l M$ denotes a closed smooth differential forms on M with the index indicating its degree, and $[\omega_l]$ is the corresponding de Rham cohomology class.
10) Let $\theta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Compute the Lie derivative $\mathcal{L}_X \theta$.