Geometry/Topology Qualifying Exam, August, 2014
Department of Mathematics & Statistics
University of New Mexico

Instructions: Do the following problems. Show all your work.

1) Let $f : X \to Y$ be a continuous surjection between two topological spaces. Show that the image $f(A)$ of an everywhere dense set $A \subset X$ is an everywhere dense set.

2) Show that every (topological) covering map $p : Y \to X$ between two topological spaces X and Y is an open map.

3) Let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map and γ a loop at a. Show that the (unique) lift $\tilde{\gamma}$ of γ starting at $\tilde{a}$ is a loop at $\tilde{a}$ if and only if $[\gamma] \in p_* \Big(\pi_1(\tilde{X}, \tilde{a})\Big)$.

4) Find the fundamental group of the cylinder $Y = \{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq 1\} \subset \mathbb{R}^3$.

5) Let $\pi : E \to M$ be a smooth rank k vector bundle. Show that any section $f \in \Gamma(E)$ is a smooth embedding of M in the total space E.

6) Suppose $\mathbb{Z}$ acts on $\mathbb{R}^2$ via $n \circ (x, y) = A^n \cdot (x, y)^t$, where
$$\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}$$
Is the quotient map $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}$ a covering map?

7) Suppose $\mathbb{Z}$ acts on $\mathbb{R}^2$ via $n \circ (x, y) = A^n \cdot (x, y)^t$, where
$$\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}$$
Is the quotient space $\mathbb{R}^2/\mathbb{Z}$ a Hausdorff space?

8) Show that the tangent space of $SL(n)$ at the identity matrix $\mathbb{I}$ is given by
$$T_{\mathbb{I}} SL(n) = \{B \in M(n) \mid \operatorname{tr} B = 0\}$$
using $SL(n) \hookrightarrow M(n) \cong \mathbb{R}^{n^2}$. Hint: Take a curve $A(t) \in SL(n)$ with $A(0) = \mathbb{I}$, $A'(0) = B$ and prove that $\det_{*A}(B) = (\det A) \operatorname{tr}(A^{-1}B)$.

9) Let $\theta = x dy \wedge dz - y dz \wedge dx + z^2 dx \wedge dy$ and $X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$. Compute the Lie derivative $\mathcal{L}_X \theta$.

10) Let ω is an exterior $k-$form on a smooth manifold M, $\omega \in \Lambda^k M$. Show that if k is odd then $\omega \wedge \omega = 0$.