Geometry/Topology Qualifying Exam, August, 2016
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Instructions: Do the following problems. Show all your work.

1) Show that a smooth submersion between two smooth manifolds is an open map, i.e., if $f : M \to N$ is a smooth map, which is a submersion, then f is an open map.
2) Let X be a Hausdorff space and f a continuous map $f : X \to X$. Show that the fixed point set of f is closed in X, i.e., the set $C = \{x \in X \mid f(x) = x\}$ is closed in X.
3) Let $p : X \to Y$ be a covering map where Y is simply connected and X is path connected. Show that p is a homeomorphism.
4) Suppose $f_i : X \to Y$, $i = 1, 2$, are two continuous maps that are homotopy equivalent. Show that the number of path-connected components of Y containing images of path-connected components X under the maps f_i coincide. In other words, the images of the path-connected components of X under both maps f_i will "hit" the same number of path-connected components (in fact same path-connected components) of Y.
5) For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by
$$\theta(t, z) = e^{it} z.$$
Show that the infinitesimal generator of θ is a smooth nonvanishing vector field on $\mathbb{S}^{2n-1}$.
6) Consider the following smooth 1-form on $\mathbb{R}^3$:
$$\eta = -\frac{4xzdx}{(x^2+1)^2} + \frac{2ydy}{y^2+1} + \frac{2dz}{x^2+1}$$
   (a) Show that η is an exact form.
   (b) Evaluate the (line) integral of η along the one-dimensional manifold with boundary, which is the straight line segment from $(0, 0, 0)$ to $(1, 1, 1)$.
7) Define a smooth 2-form on $\mathbb{R}^3$ by
$$\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy.$$
   (a) Compute ω in spherical coordinate $(\rho,\ \varphi,\ \theta)$ defined by
$$(x,\ y,\ z) = (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi).$$
   (b) Compute the pullback $\iota_{\mathbb{S}}^* \omega$ to $\mathbb{S}^2$, using coordinates $(\varphi,\ \theta)$ on the open subset where these coordinates are defined.
   (c) Show that $\iota_{\mathbb{S}}^*\ \omega$ is nowhere zero.
8) For the following pair of smooth vector fields X, Y in $\mathbb{R}^3$, compute the Lie derivative $\mathcal{L}_X Y,\ \mathcal{L}_Y X$.

$$X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}; \quad Y = \frac{\partial}{\partial y}.$$

9) Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the smooth map $F(x, y) = [x, y, 1]$, and let

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

be a smooth vector field in the two-dimensional real projective space $\mathbb{R}^2$. Prove that there is a smooth vector field Y in $\mathbb{RP}^2$ that is F-related to X, and compute its coordinate representation in terms of the following coordinate charts:

$$\begin{aligned} U_0 =& \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_0 \neq 0.\} \\ U_1 =& \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_1 \neq 0.\} \\ U_2 =& \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_2 \neq 0.\} \end{aligned}$$

10) a) Find the fundamental group of the real projective space $\mathbb{R}P^2$. b) Find the fundamental group of the real projective space $\mathbb{R}P^2$ with one point removed. Note: you are free to use any of definitions of the real projective space.