Geometry/Topology Qualifying Exam, Fall, 2018
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Instructions: Please try all of the 10 problems on the exam. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. *You can work on any of the parts of the given problems even if you skip some of the other parts.* Clear and concise answers with good justification will improve your score.

1) Give the definition of a normal topological space. Show that a smooth manifold is a normal topological space.

2) Show that the tangent bundle to a Lie group is a trivial vector bundle.

3) Prove that the product of a finite number of compact metric spaces is compact. Note: state clearly which definition of compactness you are using.

4) Give the definition of a metric space. Prove that metrizability is a topological property.

5) Show that the map $p : \mathbb{C} \to \mathbb{C}\backslash\{0\}$, defined by
$$p(z) = e^z, \quad z \in \mathbb{C},$$
is a covering map.

6) Let S^1 be the unit circle centered at the origin considered as a submanifold of $\mathbb{R}^2$ and T be the quotient manifold $\mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integer numbers acting on $\mathbb{R}$ by addition. Let $\pi : \mathbb{R} \to T$ be the quotient map and $j : T \to S^1 \subset \mathbb{R}^2$ be the map defined by
$$j\left(\pi\left(t\right)\right) = \left(\cos(2\pi t), \sin(2\pi t)\right).$$
   a) Show that the map j is a diffeomorphism.
   b) Let $\Omega = -ydx + xdy$ on $\mathbb{R}^2$ and ω be the restriction of Ω to the unit circle, i.e,
   $$\omega = \Omega|_{S^1} \stackrel{def}{=} i^*\Omega,$$
   where i is the identity (inclusion) map, $i : S^1 \to \mathbb{R}^2$. Show that ω is closed but not exact differential form.
   c) Compute $(j \circ \pi)^*\omega$ and $\int_T j^*\omega$. Determine the cohomology groups of T.

7) Let $\theta = xdy \wedge dz - ydz \wedge dx - zdx \wedge dy$ and $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Compute the Lie derivative $\mathcal{L}_X\theta$.

8) Let $\Omega = dx^1 \wedge dx^2 \wedge dx^3$ be the volume form on $M = \mathbb{R}^3$.

a) Given a smooth vector field $X = f_1 \overrightarrow{i} + f_2 \overrightarrow{j} + 3_3 \overrightarrow{k}$ on $\mathbb{R}^3$ show that the interior product (contraction) of Ω and X is given by

$$\iota_X \Omega = f_1 dx^2 \wedge dx^3 + f_3 dx^1 \wedge dx^2 + f_3 dx^1 \wedge dx^2.$$

b) For a smooth vector field X on $\mathbb{R}^3$ let $div\, X$ be the function defined by Lie derivative

$$\mathcal{L}_X \Omega = (div X)\Omega.$$

By using Cartan's formula, show that $div X$ is the usual divergence.

9) Let $\mathrm{SL}(2 : \mathbb{R})$ be the set of all 2×2 matrices (with real entries) with determinant equal to one.

a) Show that $\mathrm{SL}(2 : \mathbb{R})$ is a submanifold of the vector space $M_2(\mathbb{R})$ of all 2×2 matrices with real entries.

b) Describe in terms of matrices the tangent space to $\mathrm{SL}(2 : \mathbb{R})$ at the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What is the dimension of $\mathrm{SL}(2 : \mathbb{R})$?

c) For $t \in \mathbb{R}$ and $A \in \mathrm{SL}(2 : \mathbb{R})$ let

$$\phi_t(A) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} A.$$

Show that the above construction defines an action of the additive group $\mathbb{R}$ on $\mathrm{SL}(2 : \mathbb{R})$ and a corresponding 1-parameter group of diffeomorphisms of $\mathrm{SL}(2 : \mathbb{R})$. Find the corresponding infinitesimal generator.

10) Let $M = \{(x_1, y_1, x_2, y_2) \mid x_1^2 + y_1^2 = 1,\ x_1^2 + y_1^2 = 1\}$.

a) Show that M is a submanifold of $\mathbb{R}^4$.

b) Show M is a diffeomorphic to a certain well known compact submanifold of $\mathbb{R}^3$.Note: you should determine the manifold in question.