Geometry/Topology Qualifying Exam, January, 2019
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Instructions: Please hand in all solutions to all of the following **10** problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1) Consider the subset $X = [0,1] \cup \{-2\}$ of $\mathbb{R}$ but, instead of the usual topology, take the topology that has for a base all sets of the form four types: (a,b) where $0 < a < b < 1$; or $(a,1]$ where $0 < a < 1$; or $[0,b)$ where $0 < b < 1$; or $\{-2\} \cup (0,b)$ where $0 < b < 1$. Show that every continuous function from X to $\mathbb{R}^2$, using the usual topology on $\mathbb{R}^2$, is bounded.

2) Let $X = \mathbb{D} \cup ([2,3] \times \{0\})$ with the relative topology taken from the usual topology on the plane. Here $\mathbb{D}$ is the closed unit disk at the origin of the plane. Define $\sim$, an equivalence relation on X, by $\boldsymbol{p} \sim \boldsymbol{p}$ for all $\boldsymbol{p}$ in X as well as $(x,y) \sim (2,0)$ and $(2,0) \sim (x,y)$ whenever $x^2 + y^2 = 1$. Let $Y = X/\sim$ and let $p : X \to Y$ denote the quotient map. Show that Y is not homeomorphic to the standard unit sphere in three-space.

3) Suppose X and Y are homotopy equivalent. Show that if X is connected then Y is connected.

4) Suppose X is a Hausdorff topological space. Let Y denote the quotient space of $X \times [-1,1]$ by the relation $\sim$ which we define by

$$(x,t) \sim (x,t) \quad \forall x \in X,\, \forall t \in [-1,1]$$

and

$$(x,s) \sim (y,t) \quad \forall x,y \in X,\, \forall s,t \in \{-1,1\}.$$

Prove that Y is Hausdorff.

5) Suppose Y is a retract of X and that y_0 is a selected element of Y. Show that if the group $\pi_1(X, y_0)$ is abelian then $\pi(Y, y_0)$ is abelian.

6) Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. We define an equivalence relation $\sim$ on S^2 by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^2.$$

Prove that the induced map $f : H^2/\sim \;\to\; S^2/\sim$ is a homeomorphism.

7) If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M, show that
$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

8) Show that the tangent space at the identity I of the unitary group $U(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

9) Let $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ and $X = -y\partial/\partial x + x\partial/\partial y$ on the unit 2-sphere S^2 in $\mathbb{R}^3$. Compute the Lie derivative $\mathscr{L}_X\omega$.

10) Suppose N and M are connected, oriented n-manifolds and $F : N \to M$ is a diffeomorphism. Prove that for any $\omega \in \Omega_c^n(M)$,
$$\int_N F^*\omega = \pm \int_M \omega,$$
where the sign depends on whether F is orientation-preserving or orientation-reversing.