Geometry/Topology Qualifying Exam, August, 2019
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Instructions: Please try all of the 10 problems on the exam. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. *You can work on any of the parts of the given problems even if you skip some of the other parts.* Clear and concise answers with good justification will improve your score.

1. Show that the unit ball in $\mathbb{R}^n$ (the set of points whose coordinates satisfy $x_1^2 + \cdots + x_n^2 \leq 1$) and the unit cube (points whose coordinates satisfy $|x_i| \leq 1,\ 1 \leq i \leq n$) are homeomorphic if they are both given the subspace topology from $\mathbb{R}^n$.
2. Show that an infinite subset of a closed bounded set in $\mathbb{R}$ must have a limit point.
3. Find a topological space and a compact subset whose closure is not compact.
4. Let $\mathcal{C}$ denote the unit circle in the plane. Suppose $f : \mathcal{C} \to \mathcal{C}$ is a map which is not homotopic to the identity. Prove that $f(x) = -x$ for some point x of $\mathcal{C}$.
5. a) Show that $\mathbb{R}^n - \{0\}$ is homotopy equivalent to the unit sphere in $\mathbb{R}^n$ (the set of points whose coordinates satisfy $x_1^2 + \cdots + x_n^2 = 1$).
   b) Determine $\pi_1(\mathbb{R}^n \backslash \{0\})$.
6. a) Let E be a smooth vector bundle E over a smooth differentiable manifold M without boundary. Show that every section $f : M \to E$ is an embedding of M.
   b) Let G be a Lie group acting smoothly on a smooth manifold M without boundary. Show that each orbit is an immersed submanifold.
7. Let $X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$ and $\omega = \sum_{j=1}^n (-1)^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$ in $\mathbb{R}^n$.
   a) Determine the flow generated by X.
   b) Compute $i_X\, \omega$.
   c) Compute $\mathcal{L}_X \omega$.
   d) Compute
   $$\int_{S^{n-1}} \omega,$$
   where S^{n-1} is the $(n-1)-$dimensional unit sphere in $\mathbb{R}^n$.
8. Let α be a closed 2-form on S^4, where S^4 is the four dimensional unit sphere in $\mathbb{R}^5$.
   a) Show that $[\alpha \wedge \alpha] = 0$ in $H^4(S^4)$.
   b) Show that
   $$\int_{S^4} \alpha \wedge \alpha = 0.$$
   c) Show that $\alpha \wedge \alpha$ is not a volume form on S^4.

9. Let $\theta = \sin y dx + \cos x dz$ on $\mathbb{R}^3$ and $H = Ker\,\theta \subset T\mathbb{R}^3$.
   a) Show that H is $2-$dimensinal vector sub-bundle of TM that is non-integrable.
   b) Find the unique vector field ξ on $\mathbb{R}^3$ such that
   $$i_\xi d\theta = 0 \quad \text{and} \quad \theta(\xi) = 1.$$
   c) Show that the flow of ξ preserves θ.
   d) Consider $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, where the action of $(k, l, m) \in \mathbb{Z}^3$ is given by
   $$(k, l, m) \cdot (x, y, z) = (x + k2\pi, y + l2\pi, z + m2\pi).$$
   Show that there exists $\eta \in \Omega^1 T^3$ such that $\pi^*\eta = \theta$, where $\pi : \mathbb{R}^3 \to T^3$ is the quotient map.
10. Consider the action of S^1 on $\mathbb{R}^3$ given by $\zeta \cdot (z, t) \mapsto (\zeta z, t)$ where $\zeta \in S^1$, $z = x + iy \in \mathbb{C}$ is identified with $(x, y) \in \mathbb{R}^2$ in the usual way, and $t \in \mathbb{R}$.
   a) Show that the quotient space $\mathbb{R}^3/S^1$ is not a smooth differentiable manifold without boundary.
   b) Let M be $\mathbb{R}^3$ minus the $x-$axis. Show that the quotient space M/S^1 is a smooth differentiable manifold without boundary.