Geometry/Topology Qualifying Exam, January, 2020
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Instructions:

- You have 3 hours to complete the 10 exam prolems. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

1. Let $I = [0, 1]$ and $Q = I^2$. Suppose that $f : I \to Q$ is a continuous surjective map.
   a) Show that f cannot be injective.
   b) Let $S = \{p \in Q \mid f^{-1}(p) \text{ is not a singleton } \}$. Show that S is dense in Q.
2. For a subset A in the metric space (X, d) let
   $$d_A(x) = \inf\{d(x, a) \mid a \in A\}$$
   be the distance from a point x to A.
   a) Show that d_A is a Lipschitz continuous function on X.
   b) Show that if F is a compact subset of X and $x \in X$, then there exists $a \in F$, such that,
   $$d(x, a) = d_F(x).$$
   c) Show that if F_1 and F_2 are two disjoint closed subsets of X, then the function
   $$\phi(x) = \frac{d_{F_1}(x)}{d_{F_1}(x) + d_{F_2}(x)}$$
   satisfies the claim of Urysohn's lemma.
   d) Show that every metric space is a T_4 space.
3. Let X be a path-connected and locally path-connected space, such that, $\pi_1(X)$ is finite. Let $f : X \to S^1$ be a continuous function, where S^1 is the unit circle in the plane.
   a) Show that there is a continuous lift $\tilde{f}$ of f to the universal cover of the unit circle S^1.
   b) Show that f is null-homotopic, i.e., f is homotopic to a constant map.
   c) Let $\sigma : S^1 \to S^1$ be the map $\sigma(x) = -x$. Show that σ is homotopic to the identity.
4. Suppose the unit two-dimensional sphere S^2 is covered by the union of three closed sets F_j, $j = 1, 2, 3$,
   $$S^2 = F_1 \cup F_2 \cup F_3.$$
   Define the function $F : S^2 \to \mathbb{R}^2$ by letting
   $$F(x) = (f_1(x), f_2(x)),$$

where

$$f_i(x) = \inf\{\|x-y\| \mid y \in F_i\}, \quad i = 1, 2,$$

and $\|x-y\|$ denotes the Euclidean distance.

a) Show that there exists $x_0 \in S^2$ such that $F(x_0) = F(-x_0)$.

b) Show that at least one of the sets F_1, F_2 and F_3 contains two antipodal points. Hint: consider the point x_0 from part b).

5. Let p and q be relatively prime positive integers satisfying $1 \leq q < p$. Let $\zeta = e^{i2\pi/p}$ be a generator of the group of the p-th roots of unity C_p. Let S^3 be the unit three dimensional sphere described using complex coordinates as

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}.$$

a) Show that the map $\theta : C_p \times S^3 \to S^3$,

$$\theta(\zeta^k, (z, w)) = (\zeta^k z, \zeta^{qk} w)$$

defines a smooth action of the group C_p.

b) Show that the quotient space $L(p, q) = S^3/C_p$ is a manifold.

c) Show that if $L(p, q)$ and $L(p', q')$ are diffeomorphic (homeomorphic will suffice) then $p = p'$.

6. Let G be a Lie group acting smoothly on a smooth manifold M.

a) Show that each orbit is an immersed submanifold, which is embedded if the action is proper.

b) Let $G = \mathbb{R}^*$, the multiplicative group of non-zero real numbers, and $M = \mathbb{RP}^1$. We define the action of G on M in the following way: for any $g \in G$, $p = [x_1, x_2] \in M$, $g{\cdot}p = [gx_1, x_2]$. List all the orbits and determine whether they are embedded submanifolds.

7. Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F(x^1, \dots, x^n) = [x^1, \dots, x^n, 1]$, where $[x^1, \dots, x^n, 1]$ is the point in the real projective space $\mathbb{RP}^n$ representing the line defined by $(x^1, \dots, x^n, 1)$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$.

8. Let M be a smooth manifold, V be a smooth vector field on M and ω be a smooth differential form on M. Prove that

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega).$$

9. Prove that if G is an abelian Lie group, then $Lie(G)$ is abelian.

10. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w, x, y, z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = xyz\ dw \wedge dy.$$