Geometry/Topology Qualifying Exam, August, 2020
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Instructions:

- You have 3 hours to complete the 10 exam prolems. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

1. a) Show that if A is a connected subspace of X, then the closure $\bar{A}$ is connected as well.
   b) Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ equipped with the subspace topology.
      i) Is $\mathbb{Q}$ a connected topological space?
      ii) Is $\mathbb{Q}$ a topological manifold?

2. Consider the map $q : S^1 \times [0,1] \to \overline{\mathbb{D}}$ defined by
$$q(\zeta, t) = (1-t)\zeta,$$
where $\overline{\mathbb{D}}$ is the closed unit disc in the plane with boundary S^1. Show that q is a quotient map.

3. Let $n > 1$ and S^n be the unit n-dimensional Euclidean sphere in R^{n+1}.
   a) Show that there is no smooth surjective map $\gamma : S^1 \to S^n$.
   b) Prove that for $n > 1$ the unit n-dimensional Euclidean sphere S^n in R^{n+1} has a trivial fundamental group.

4. Let S^1 be the unit Euclidean circle in the xy-plane $\mathbb{R}^2$ and
$$C = S^1 \times (-1,1) = \{(x,y,z) \mid x^2 + y^2 = 1, -1 < z < 1\}.$$
   a) Let α be a 1-form on C which defines the cohomology class $[\alpha] \in H^1(C)$-the first cohomology class of C. Show that $[\alpha] = 0$ if and only if
$$\int_S \alpha = 0,$$
   where S is the embedded submanifold given by $i : S^1 \times \{0\} \hookrightarrow C$,
$$i(p) = (p, 0) \in S^1 \times (-1,1), \qquad p \in S^1.$$
   b) Determine if α is an exact form on C for
$$\alpha = \frac{1}{x^2+y^2}\,(xdy - ydx) + x^2 ydz|_C.$$

5. Show that the map $g : S^1 \times S^1 \to S^3$, $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, given by
$$g(e^{is}, e^{it}) = \frac{1}{\sqrt{2}}\left(e^{is}, e^{it}\right) \in \mathbb{C}^2$$
is a smooth embedding. Here, S^n denotes the $n-$dimensional Euclidean unit sphere in $\mathbb{R}^{n+1}$.

6. Let X be a complete metric space with the metric d. Let $f : X \to X$ be a map such that $d(f(a), f(b)) < \lambda d(a, b)$ for any $a, b \in X$ with $a \neq b$ and $0 < \lambda < 1$. Show that f has a unique fixed point.

7. Let $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ be the the upper half-plane, let $a, b, c, d \in \mathbb{R}$, and let
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0.$$
Prove that
$$\begin{aligned} f : \mathcal{H} &\to \mathcal{H} \\ z &\mapsto \frac{az+b}{cz+d} \end{aligned}$$
is a homeomorphism.

8. Let $G = \{X \in GL(n, 3) \mid X^t \eta X = \eta,\ det(X) = 1\}$, where
$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
   (a) Show that G is a Lie subgroup of $GL(n, 3)$.
   (b) Calculate the Lie algebra of G.

9. Let M be the set of $n \times n$ symmetric matrices. Let $F : GL(n, \mathbb{R}) \to M$ be given by $F(A) = A^t A$. Show that
   (a) M is a n(n+1)/2 dimensional submanifold of R^{n^2}.
   (b) $F(I_n)$ is a regular value of F.

10. Let $S^n \subset \mathbb{R}^{n+1}$ be the $n-$dimensional Euclidean unit sphere in $\mathbb{R}^{n+1}$. Show that the normal bundle of S^n in $\mathbb{R}^{n+1}$ is trivial. Here, the normal bundle is defined as the sub-bundle of $T\mathbb{R}^{n+1}$ consisting of all vectors that are (Euclidean) orthogonal to $TS^n \subset T\mathbb{R}^{n+1}$.