Geometry/Topology Qualifying Exam, January, 2021
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Instructions:

- You have 3 hours to complete the exam. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

1. Let X be a topological space and Y be a compact Hausdorff space. Show that $f : X \to Y$ is continuous if and only if the graph of f,
$$G_f = \{(x, f(x)) \mid x \in X\}$$
is closed in $X \times Y$.
2. Let X be a regular topological space with a countable basis; let U be an open subset in X.
   (a) Show that U equals a countable union of closed sets of X.
   (b) Show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.
3. Let X be the quotient space obtained from the unit ball B^2 in $\mathbb{R}^2$ by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to the projective plane $\mathbb{RP}^2$.
4. Let $q : E \to X$ be a covering map with $q^{-1}(x)$ finite and nonempty for all $x \in X$. Show that E is compact if and only if X is compact.
5. Show that the solution set S of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a manifold of dimension 2.
6. Show that the orthogonal group $O(n)$ is compact by proving the following two statements:
   (a) $O(n)$ is a closed subset of $\mathbb{R}^{n\times n}$.
   (b) $O(n)$ is a bounded subset of $\mathbb{R}^{n\times n}$.
7. Let K be the product manifold $\mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n$. We define a group structure on K by setting $(g_1, x_1)(g_2, x_2) = (g_1 \cdot g_2, g_1 \cdot x_2 + x_1)$. Show that K is a Lie group.
8. Prove that the group $\mathrm{SL}(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{z = x + yi | y > 0\}$ in $\mathbb{C}$ properly and discontinuous. More precisely, prove the following statements.

(a) The map $z \mapsto \gamma(z) = \frac{az+b}{cz+d}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ is a homeomorphism, and

(b) For every $\tau \in \mathbb{H}$, there exits a neighborhood $U \subset \mathbb{H}$ of τ such that

$$U \cap \gamma(U) \neq \emptyset \qquad \text{if and only if} \qquad \gamma = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

9. Consider the 2-form

$$\omega = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2$$

on $\mathbb{R}^3 \backslash \{0\}$.

(a) Let $i : \mathbb{S}^2 \to \mathbb{R}^3 \backslash \{0\}$ be the inclusion map, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, calculate

$$\int_{\mathbb{S}^2} i^* \omega.$$

(b) Determine values for k such that the 2-form $(x_1^2 + x_2^2 + x_3^2)^k \omega$ is exact.

(c) Determine values for k such that the 2-form $(x_1^2 + x_2^2 + x_3^2)^k \omega$ is closed.

10. Let M be a Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth function. Define a vector field $\nabla f = \text{grad } f$ in M as

$$< \nabla f, v >= df(v)$$

for all vector field v in M. Show that ∇f is smooth.