Geometry/Topology Qualifying Exam, August, 2021
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**Instructions**:

- You have 3 hours to complete the exam. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

(1) Let $X$ be an infinite set equipped with the co-finite topology, i.e., a set different from the empty set or the whole space is open if and only if its complement is finite.
  a) Show that this is indeed a topology on the set $X$.
  b) Show that every sequence $\{x_n\}$ of *different* elements of $X$ is convergent and determine its possible limit(s).
  c) Is the cofinite topology Hausdorff?

(2) Let $X$ be a compact topological space and $Y$ a Hausdorff topological space. Prove that if $f : X \to Y$ is a continuous bijection, then $f$ is a homeomorphism.

(3) Let $\mathbb{RP}^2$ denote the real projective plane and $\mathbb{S}^1$ the unit circle. Prove that any continuous map $f : \mathbb{RP}^2 \to \mathbb{S}^1$ is null-homotopic, i.e., homotopic to a constant map.

(4) Let $f$ be a smooth function on the smooth manifold $M$ without boundary. Show that if $f$ achieves a local minimum at the point $p \in M$ then the differential $df_p$ vanishes, i.e., it annihilates the tangent space $T_pM$.

(5) Let $M$ be a three dimensional smooth differentiable manifold without boundary and $\theta \in \Omega^1(M)$ be a 1-from, such that, $\theta \wedge d\theta \neq 0$ everywhere.
  a) Let $H = \{X_p \in T_pM \mid p \in M, \theta(X_p) = 0\} \subset TM$ be the kernel of $\theta$. Show that $H$ is a $2-$dimensinal vector sub-bundle of $TM$ that is non-integrable.
  b) Show that there exists a unique vector $\xi \in TM$ such that

$$i_\xi d\theta = 0 \quad \text{and} \quad \theta(\xi) = 1.$$

  c) Show that the flow of $\xi$ preserves $\theta$.

(6) On $\mathbb{R}^4 \backslash \{\mathbf{0}\}$, consider the following 2-form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

  a) Write $\omega$ in polar coordinates $(r_1, \theta_1, r_2, \theta_2)$, where

$$(x_1, y_1, x_2, y_2) = (r_1 \cos\theta_1, r_1 \sin\theta_1, r_2 \cos\theta_2, r_2 \sin\theta_2).$$

b) Let $X = \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}$, calculate the contraction (interior product)

$$\iota_X \omega.$$

(7) Recall that the Poincaré lemma implies

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Consider the following differential forms on $\mathbb{R}^4$ in Cartesian coordinates $(x, y, z, w)$

$$\omega = -\frac{4xz}{(x^2+1)^2} dx + \frac{2}{x^2+1} dz + y dw,$$

$$\eta = x dy \wedge dz \wedge dw + y dz \wedge dw \wedge dx + z dw \wedge dx \wedge dy + w dx \wedge dy \wedge dz,$$

Are there $f \in C^\infty(\mathbb{R}^4)$ and $\alpha \in \Omega^2(\mathbb{R}^4)$ (i.e. $\alpha$ is a smooth 2-form on $\mathbb{R}^4$) such that $df = \omega$ and $d\alpha = \eta$? Prove your conclusion.

(8) Let $G = \mathbb{R}^*$, the multiplicative group of non-zero real numbers, and $M = \mathbb{RP}^1$ the real projective line. We define the "action" of $G$ on $M$ in the following way: for any $g \in G,\ p = [x_1, x_2] \in M,\ g \cdot p = [gx_1, x_2]$.

a) Show that this is indeed a smooth group action.

b) Is the action free? Is it proper?

c) List all the orbits and determine whether they are embedded submanifolds.

d) Is the quotient space $M/G$ a smooth manifold?

(9) Consider the set of invertible matrices $G = GL(3, \mathbb{R})$ as a a subset of $\mathbb{R}^{3\times 3} \equiv \mathbb{R}^9$.

a) Show that $G$ is a smooth embedded submanifold of $\mathbb{R}^{3\times 3}$.

b) Show that $G$ is a Lie group under the matrix multiplication.

c) Show that

$$\begin{array}{rccc} \varphi: & GL(3,\mathbb{R}) & \to & SL(3,\mathbb{R}) \times \mathbb{R}^* \\ & M & \mapsto & \left( (\det M)^{\frac{1}{3}} M, \det M \right), \end{array}$$

is a Lie group isomorphism, where $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$ is the multiplicative group of positive real numbers and $SL(3, \mathbb{R})$ is the special linear group of 3-by-3 matrices of determinant 1.

(10) Let $\Omega$ be a volume form on an oriented smooth manifold $M$ with boundary $\partial M$. For a smooth vector field $X$ let $div\, X$ be the function defined by $\mathcal{L}_X \Omega = (div X)\Omega$, where $\mathcal{L}_X$ is the Lie derivative. Show that if $M$ is a compact then

$$\int_M div X\, \Omega = \int_{\partial M} i_X \Omega.$$