Geometry/Topology Qualifying Exam, January, 2022
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Instructions:

- You have 3 hours to complete the exam. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

(1) Let X be a compact Hausdorff space and f a continuous map $f : X \to X$. Show that the graph of f is compact in $X \times X$, i.e., the set $G = \{(x, y) \in X \times X | f(x) = y\}$ is compact in $X \times X$.

(2) Let $X = [0, 1] \times [0, 1]$ and

$$Y = \{(tx, t) \in \mathbb{R}^2 : x \in \mathbb{Q} \cap [0, 1], t \in [0, 1]\},$$

each endowed with the topology inherited from the Euclidean topology. Show that X and Y are homotopy equivalent.

(3) Compute the fundamental group of real projective plane minus one point.

(4) Let X be the topological subspace of $\mathbb{R}^3$ consisting of all solutions to $x^2z + y^2z = 0$. Determine if X is, or is not, a topological manifold. Either way, prove your answer.

(5) Suppose that X and Y are connected topological spaces. Prove that $X \times Y$ is also connected.

(6) Let M be a smooth manifold and H be a subbundle of the tangent bundle TM. Let $\pi : TM \to M$ and $\alpha : TM \to TM/H$ be the natural projections. Show that for $p \in M$ and any smooth vector fields X and Y on M, the vector $\alpha([X, Y])(p)$ depends only on the values of X and Y at p. Here, the commutator of two vector fields X and Y on M is the vector field $[X, Y]f = XYf - YXf$.

(7) Let $GL(n)$ be the general linear Lie group of $n \times n$ invertible matrices with entries real numbers with group operation given by matrix multiplication.
a) Show that the set $SO(n)$ of all real orthogonal $n \times n$ matrices of determinant equal to one is a Lie subgroup of $GL(n)$. What is the dimension of $SO(n)$?

b) Describe in terms of matrices the Lie algebra $so(n)$ of the Lie group $SO(n)$.

(8) Let M and N be smooth manifolds.
a) Show that a smooth submersion $f : M \to N$ is an open map.
b) Show that a surjective smooth submersion $f : M \to N$ is an quotient map.

(9) Let Ω be a volume form on an oriented via Ω smooth differentiable manifold M. For a smooth vector field X let $div\, X$ be the function defined by $\mathcal{L}_X\Omega = (divX)\Omega$.
a) Show that if $M = \mathbb{R}^m$ with volume form $\Omega = dx^1 \wedge \cdots \wedge dx^m$ and $X = \sum_{i=1}^m a^i(x)\frac{\partial}{\partial x^i}$, then

$$divX = \sum_{j=1}^{m} \frac{\partial a^i(x)}{\partial x^i}.$$

b) Show that if M is compact with (possibly empty) boundary ∂M, then

$$\int_M (divX)\,\Omega = \int_{\partial M} i_X\Omega.$$

Here, $i_X\Omega$ is the interior product of X and Ω.

(10) Let X be a smooth compactly supported vector field on $\mathbb{R}^{2n}$ and ω be the 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

Show that ω is invariant under the flow of X if and only if for every point $p_0 \in \mathbb{R}^{2n}$ there exists an open neighbourhood U of p_0 and a smooth function $f : U \to \mathbb{R}$ such that on U we have

$$i_X\omega = -df(X).$$

Here, $i_X\omega$ is the interior product of X and ω.