Geometry/Topology Qualifying Exam, August, 2022
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Instructions:

- You have 3 hours to complete the exam. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

(1) Let X be a compact space. Is the graph of the identity map $i : X \to X$ compact in the topological product $X \times X$? Justify your answer.

(2) Let (X, d) be a metric space. For $x_0 \in X$ and $R > 0$ let $B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}$ be the open metric ball centered at x and radius R. Let $f : X \to \mathbb{R}$ a continuous function.

a) Is it true that $\sup_{y \in B_R(x_0)} |f(y)| < \infty$? Justify your answer.

b) Is it true that

$$\sup_{y \in \bar{B}_R(x_0)} |f(y)| < \infty,$$

where $\bar{B}_r(x_0) = \{y \in X \mid d(x_0, y) \leq R\}$? Justify your answer.

c) Show that for every $x \in B_R(x_0)$ there exists an open ball $B_r(x) \subset B_R(x_0)$, such that,

$$\sup_{y \in B_r(x)} |f(y)| < \infty.$$

(3) Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Show that $\mathcal{C}$ is a base for the topology of X.

(4) Prove that there is no retraction $r : \mathbb{B}^2 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is the boundary of the closed unit disc $\mathbb{B}^2$ in $\mathbb{R}^2$

(5) A Lindelöf space is a topological space for which every open covering contains a countable subcovering. Show that if X is a Lindelöf space and Y is a compact topological space, then $X \times Y$ is Lindelöf.

(6) Let $p : Y \to B$ be a smooth submersion between two smooth manifolds without boundaries. Let $f : X \to B$ be a smooth map from the smooth manifolds without boundary X to B. Show that the fiber product

$$Y \times_B X = \{(y, x) \in Y \times X \mid p(y) = f(x)\}$$

is an embedded submanifold of the product manifold $Y \times X$.

(7) Let $M = f^{-1}(1)$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is the function $f(x, y, z) = x^2 + y^2 - z^2$. Let S be the sphere of radius 2 centered at the origin of $\mathbb{R}^3$.
a) Show that M and S are transverse, $M \pitchfork S$.
b) Determine if $M \cap S$ a submanifold of $\mathbb{R}^3$?

(8) Let M be a smooth manifold and TM its tangent bundle.
a) Show that for every for $t \in \mathbb{R}$ and $X \in TM$ the map $\phi_t : TM \to TM$ defined by $\phi_t(X) \overset{def}{=} e^t X$ is a smooth vector bundle morphism.
b) Find the local expression of the following vector field $\xi(X) = \frac{d}{dt}(e^t X)\Big|_{t=0} \in T_X TM$ on the manifold TM in the standard coordinates (x, y) of TM induced from a coordinate system x on M with y the fiber coordinate.
c) Show that for every for $t \in \mathbb{R}$ the map ϕ_t is a diffeomorphisms of the manifold TM which defines a smooth action of $\mathbb{R}$ on TM.
d) Is the above action proper?
e) Show that for each $t \in \mathbb{R}$ the vector field ξ is invariant under ϕ_t.

(9) Suppose $F : G \to H$ is a Lie group homomorphism. Show that the kernel of $F_* : Lie(G) \to Lie(H)$ is the Lie algebra of $Ker(F)$, where F_* is the Lie algebra homomorphism induced by F .

(10) Show that for any $n \geq 0,\ p \geq 1$, the de Rham cohomology group of $\mathbb{R}^n$ is

$$H^p_{dR}(\mathbb{R}^n) = 0.$$