NUMERICAL ANALYSIS QUALIFYING EXAM

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Each Problem Counts 25 Points

January 1994 **Name:** ______________________

1. (i) State the *fundamental theorem of linear algebra.*

   (ii) Find the LU decomposition of the following matrix A.

   (iii) Find bases for each of the four fundamental subspaces associated with A (that is, $\mathcal{R}(A), \mathcal{R}(A^T), \mathcal{N}(A)$, and $\mathcal{N}(A^T)$), and state the dimension of these subspaces.

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

2. (i) Describe the *singular value decomposition* of an $m \times n$ matrix A. Define all matrices that you introduce.

   (ii) For the following matrix A and vector b, find the singular value decomposition, the pseudoinverse A^+, and the minimum length least squares solution x^+ of $Ax = b$.

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}.$$

3. A real square matrix A is called positive definite symmetric (PDS) if it is symmetric, $A = A^T$, and, for any $\mathbf{x} \neq \mathbf{0}$:

$$\mathbf{x}^T A \mathbf{x} > 0.$$

   (i) Show that all eigenvalues of a PDS matrix are real and positive.

   (ii) Let $A^{(m)}$ be the $m \times m$ matrix obtained by intersecting the first m rows and columns of a PDS matrix, A. Show that $A^{(m)}$ is also PDS.

   (iii) Use the results above to show that if Gaussian elimination without pivoting is applied to a PDS matrix, only positive pivots are encountered. (Hint: consider the relationship between the pivots, the determinant, and the eigenvalues.)

   (iv) Use (iii) to prove the existence of the Cholesky decomposition of a PDS matrix: $A = LL^T$ where L is lower triangular.

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4. The power method for computing an eigenvalue of a matrix, A, is defined by:

$$y_{n+1} = Ax_n, \quad \lambda_{n+1} = y_{n+1}^T x_n, \quad x_{n+1} = \frac{y_{n+1}}{\sqrt{y_{n+1}^T y_{n+1}}},$$

where x_0 satisfying $x_0^T x_0 = 1$ is otherwise arbitrary.

(i) Show that if there is a single extreme eigenvalue, that is a simple eigenvalue λ such that $|\alpha| < |\lambda|$ for all other eigenvalues α, then the power method converges, that is $\lambda_n \to \lambda$, for most x_0. (To simplify your arguments, assume that A is diagonalizable.)

(ii) Describe the typical behavior of the method if the extreme eigenvalues correspond to a conjugate imaginary pair. In particular show that the sequence λ_n may converge to a number which is not an eigenvalue. (An example will do.)

(iii) Let A be a real skew-symmetric matrix, i.e. $A = -A^T$. Show that all eigenvalues of A are imaginary. What can you say about the eigenvalues of A^2?

(iv) Suggest a modification of the power method to compute extreme eigenvalues of a skew-symmetric matrix.