Numerical Analysis Fall 1999
MS/PhD Qualifying Examination

Instruction: Complete all four problems.

1. In this problem all matrices are real and of dimension $n \times n$.

   a) Define a normal matrix.

   b) Define an orthogonal matrix.

   c) State the properties of the eigenvalues and eigenvectors of a normal matrix? Prove your statements.

   d) Let Q denote an orthogonal matrix and let Λ denote a diagonal matrix. Is the matrix $A = Q^{-1}\Lambda Q$ always normal? (Give a proof or counterexample.)

2. Consider an overdetermined linear system

$$Ax = b$$

   where A is real of size $m \times n$, $x \in \mathbb{R}^n, b \in \mathbb{R}^m, m > n$.

   a) When is a vector $x^* \in \mathbb{R}^n$ called a least squares solution of the system $Ax = b$?

   b) Give assumptions on A so that a least squares solution exists and is uniquely determined. Prove the result.

   c) Compute the least squares solution of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} .$$

3. Let A be an $n \times n$ positive definite symmetric matrix.

   a) Prove that $A = LL^T$ where L is lower triangular.

   b) Estimate, as a function of n, the number of arithmetic operations required to compute L by elimination.

   c) Using the matrix norm subordinate to the standard Euclidean vector norm, relate $\|L\|$ to $\|A\|$. Does this have any implications for the numerical stability of the computation of L?

4. Suppose A is an $n \times n$ matrix with a known LU factorization and let B be a rank-one perturbation of A, i.e.

$$B = A + uv^T.$$

   Assuming both A and B are invertible, describe how to compute the solution to:

$$Bx = c$$

   in $O(n^2)$ operations.