Numerical Analysis Spring 2000
MS/PhD Qualifying Examination

Instructions: Complete all four problems.

1. Determine which of the following statements are true or false. Justify your answer.

   Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ real, square, matrices with entries a_{ij}, $1 \le i \le n$, $1 \le j \le n$ and b_{ij}, $1 \le i \le n$, $1 \le j \le n$; respectively.

   (a) Suppose $\mathbf{A}$ is invertible. Then there is a number $\epsilon > 0$ such that if $\mathbf{B}$ satisfies $|b_{ij}| < \epsilon$, then $\mathbf{A} + \mathbf{B}$ is invertible.

   (b) Suppose $\mathbf{A}$ is not invertible. Then there is a number $\epsilon > 0$ such that if $|b_{ij}| < \epsilon$, then then $\mathbf{A} + \mathbf{B}$ is not invertible.

2. Let $\mathbf{V}$ be the space of polynomials of degree less than or equal to 2; *i.e.* $p \in \mathbf{V} \iff p(x) = ax^2 + bx + c$. Introduce coordinates in $\mathbf{V}$ by associating each such polynomial with the point in $\mathbf{R}^3$: $(p(-1), p(0), p(1))$.

   (a) Find the mapping of $\mathbf{R}^3 \to \mathbf{R}^3$ corresponding to differentiation in $\mathbf{V}$; (*i.e.*, the matrix representing differentiation in the chosen coordinate system.)

   (b) Find vectors in $\mathbf{R}^3$ spanning the range of this mapping. What are the corresponding elements of $\mathbf{V}$?

   (c) Find vectors in $\mathbf{R}^3$ spanning the null space of this mapping. What are the corresponding elements of $\mathbf{V}$?

3. Let A be an $m \times n$ real matrix with $m \ge n$.

   a. What is the QR factorization of A? Describe how the QR factorization can be used to find the least squares solution of $Ax = b$.

   b. Define Householder reflections or Givens rotations.

   c. Describe an algorithm for computing the QR factorization using the transformation you defined above.

   d. Estimate the complexity of the algorithm described above. Treat separately the cases where Q must be saved and cases where it isn't needed.

   e. Discuss the numerical stability of the factorization.

4. Let A be an $n \times n$ real symmetric matrix.

a. What does it mean for A to be positive definite?

b. Consider the application of Gaussian elimination without pivoting to A. Show that A is positive definite if and only if all pivots encountered are positive.

c. How can the result above be used to devise a test for positive definiteness?

d. Use the result in [b.] to prove that positive definite symmetric matrices possess a Cholesky factorization, i.e. can be written as LL^T where L is lower triangular.