Numerical Analysis Exam
August 2000

Complete all five problems. Unless otherwise stated, all matrices and vectors are real. Good luck!

1. (25 pts)
   Let $A, C \in R^{n\times n}$ with $A = A^T$ and C nonsingular.
   (a) Prove that the eigenvalues of A are real.
   Now, let A be nonsingular and let $B := C^T AC$. Prove:
   (b) B is nonsingular
   (c) A and B have the same number of positive (and negative) eigenvalues (that is, prove Sylvester's law of inertia).

2. (15 pts)
   Let
   $$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$
   (a) Find the Moore-Penrose pseudoinverse, A^+.
   (b) Using A^+ find the solution to the least squares problem $Ax = b$ where $b = (1,\ 0,\ 1)^T$.

3. (10 pts)
   Let $A \in R^{n\times n}$ be upper triangular. If $AA^T = A^T A$ prove that A is a diagonal matrix.

4. (25 pts)
   (a) Define the condition number, $\kappa(A)$ of an $n \times n$ matrix, A, and show that $\kappa(A) \geq 1$.
   (b) Show that orthogonal matrices have the minimum condition number if the Euclidean norm is used.
   (c) Let $Ax = b$, $(A + E)x = (b + e)$. Prove the fundamental inequality:
   $$\frac{\|e\|}{\|b\|} \leq \kappa(A)\frac{\|E\|}{\|A\|}.$$

5. (25 pts)
   Let A be an $n \times n$ matrix. Prove that if and only if $\rho(A) < 1$:
   $$\lim_{n\to\infty} \sum_{j=0}^{n} A^j = (I - A)^{-1}.$$
   Here $\rho(A)$ is the spectral radius of A. Relate this result to an iterative method for solving $(I - A)x = b$.