Corrected version

Numerical Analysis Exam - Fall 2002

Answer all four questions.

1. Consider a matrix $A \in \mathbf{C}^{n\times n}$, vector $x \in \mathbf{C}^n$ with $x \neq 0$. The Rayleigh Quotient, $\mathcal{R}_A(x)$, is defined as the quantity

$$\mathcal{R}_A(x) = \frac{x^H A x}{x^H x}$$

   (a) State the definition of a **hermitian** matrix and characterize its spectrum, eigenvectors and diagonalizability.

   (b) Prove that if A is *hermitian* then $\mathcal{R}_A(x)$ is real and

$$\lambda_1 \leq \mathcal{R}_A(x) \leq \lambda_n,$$

   where λ_1 (resp. λ_n) is the least (resp. greatest) eigenvalue of A.

   (c) State the definition of a **normal** matrix and characterize its spectrum, eigenvectors and diagonalizability.

   (d) Prove that if A is *normal* then $\mathcal{R}_A(x)$ is in general complex and it lies in the **Convex Hull** of the eigenvalues $\lambda_i, i = 1, \ldots, n$ of A; that is, show that if $z = \mathcal{R}_A(x)$ for some $x \in \mathbf{C}^n \setminus \{0\}$ then there exist n numbers $0 \leq c_i \leq 1$, $i = 1, \ldots, n$ with $\sum_{i=1}^n c_i = 1$ such that

$$z = \sum_{i=1}^{n} c_i \lambda_i \ .$$

2. An $n \times n$ matrix A is skew-Hermitian if $a_{ij} = -a^*_{ji}$, $1 \leq i, j \leq n$. Prove or give a counterexample to each of the following statements about A:

   (a) Is a normal matrix.

   (b) Has $-\lambda$ as an eigenvalue if λ is an eigenvalue.

   (c) Is singular if it is of odd order (i.e. n is odd).

   (d) Has pure imaginary eigenvalues.

   (e) The matrix $B = I + A$ is nonsingular.

   (f) The matrix $C = (I + A)^{-1}(I - A)$ is unitary.

3. A real, symmetric matrix, A, is positive definite if:

$$x^T A x > 0, \quad \text{for all} \quad x \in R^n, \quad x \neq 0.$$

(a) Show that if A is a positive definite symmetric matrix then if Gaussian elimination (without pivoting) is applied to A only positive pivots are encountered. Thus conclude that A has an LU factorization.

(b) Use the previous result to show that A has a Cholesky factorization, that is $A = R^T R$ where R is upper triangular.

(c) Describe an algorithm to compute the Cholesky factorization using minimal work and storage.

(d) Derive an inequality relating the 2-norms of A and R. What are the implications of this inequality for the numerical stability of the Cholesky factorization?

4. Suppose a family of matrices T_n satisfies:

$$(T_n)_{ij} = 0, \quad i > 1 \ \text{ and } \ |i - j| > 1.$$

That is, T_n is a tridiagonal matrix except for the first row which may be full. Describe in detail an algorithm for solving systems of linear equations with coefficient matrices T_n using $O(n)$ flops and storage. Under what circumstances does your algorithm break down?