Numerical Analysis Spring 2004
MS/PhD Qualifying Examination

Instructions: Complete all problems. Explain your answers.

1. Consider the subspace $M \subset \mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}.$$

   (a) Write down the matrix $\mathbf{P}$ representing the orthogonal projector onto M.

   (b) What is the null space of $\mathbf{P}$?

   (c) What is the range of $\mathbf{P}$?

   (d) Find the vector x in M that is closest in the 2-norm sense to the vector $c = (1,1,1,1)^T$.

2. Consider solving the linear system of equations, $\mathbf{A}x = b$ with $\mathbf{A}$ an $n \times n$ real matrix and x, $b \in \mathbb{R}^n$, using an iterative method based on the splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$, with $\mathbf{M}$ invertible. The iterative method is $x_{k+1} = \mathbf{G}x_k + f$ where the iteration matrix is $\mathbf{G} = \mathbf{M}^{-1}\mathbf{N}$ and $f = \mathbf{M}^{-1}b$.

   (a) What are $\mathbf{G}_J$ and $\mathbf{G}_{GS}$, the iteration matrices for the Jacobi and Gauss-Seidel iterative methods, respectively?

   (b) The iterative method is called symmetrizable if for some nonsingular matrix $\mathbf{W} \in \mathbb{R}^{n \times n}$, the matrix $\mathbf{W}(\mathbf{I} - \mathbf{G})\mathbf{W}^{-1}$ is symmetric positive definite. $\mathbf{W}$ is called the symmetrization matrix. If the iterative method is symmetrizable then prove:

      i. The eigenvalues of $\mathbf{G}$ are real.

      ii. The largest eigenvalue of $\mathbf{G}$ is less than one.

      iii. The set of eigenvectors of $\mathbf{G}$ forms a basis for $\mathbb{R}^n$.

   (c) Does a symmetrizable iterative method necessarily converge? Explain.

   (d) Show that the iterative method is symmetrizable whenever $\mathbf{A}$ and $\mathbf{M}$ are symmetric positive definite matrices. (Hint: consider $\mathbf{A}^{1/2}$.)

   (e) If $\mathbf{A}$ is symmetric positive definite, are Jacobi or Gauss-Seidel symmetrizable? Explain.

3. Let:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

be the singular value decomposition of the $m \times n$ matrix $\mathbf{A}$.

i. Supposing rank($\mathbf{A}$) $= r$, use $\mathbf{U}$ and $\mathbf{V}$ to find orthonormal bases for the column space of $\mathbf{A}$, the row space of $\mathbf{A}$, the null space of $\mathbf{A}$, and the null space of $\mathbf{A}^T$.

ii. Supposing $m = n$, use the SVD to find the Euclidean norm condition number of $\mathbf{A}$. Justify your answer.

iii. Use the SVD to find the matrix, $\mathbf{B}$, of rank $q < r$ which is closest to $\mathbf{A}$ in the Euclidean norm. Justify your answer.

iv. Let $\mathbf{W}$ be an $n \times n$ matrix. Suppose the singular values of $\mathbf{W}$ are:

$$\sigma_k = 2^{1-k}.$$

Using standard double precision arithmetic, for which $\epsilon_{\text{machine}} \approx 2^{-52}$, for n how large can one expect to accurately solve a linear system with coefficient matrix $\mathbf{W}$?

v. Consider the problem of computing $\mathbf{W}z$ using standard double precision arithmetic and $\mathbf{W}$ from the previous problem. Assuming you have precomputed the SVD, roughly how many flops should this require for $n \gg 1$? (Compute the approximate leading constant for full credit.)

4. Suppose $\mathbf{A} = \mathbf{A}^T$ is symmetric but not necessarily positive definite. Assume further that $\mathbf{A}$ has an LU factorization. Describe an algorithm for factoring A into triangular factors requiring about half the work and storage of standard Gaussian elimination. Comment on the stability of the algorithm, justifying your comments.