Numerical Analysis Fall 2008
MS/PhD Qualifying Examination

*Instructions: Write the last four digits of your SSN (***not*** *your name) on each sheet. Complete all three problems. Clear and concise answers with good justification will improve your score.*

1. (30 pts.) Let A be an $n \times n$ real matrix and let $A^{(k)}$ denote the $k \times k$ matrix formed by the intersection of the first k rows and columns of A. Prove that if $A^{(k)}$ is nonsingular, $k = 1, \dots, n$, then A has an LU factorization, where L is a unit lower triangular matrix (i.e. $l_{jj} = 1$) and U is upper triangular. Express the pivots, u_{jj}, in terms of the determinants of the $A^{(k)}$.

2. (30 pts.) Let $A = A^T$ be a real $d \times d$ symmetric matrix.

   i. Use Schur's Theorem to prove that A has real eigenvalues and can be diagonalized by an orthogonal similarity transformation, $Q^T A Q = \Lambda = \text{diag}(\lambda_i)$ where λ_i, $i = 1, \dots, n$ are the eigenvalues of A and $Q^T Q = I$.

   ii. Suppose A has exactly p positive eigenvalues. Show that there exists a p-dimensional subspace, P, of $\mathbb{R}^d$ such that $x^T A x > 0$ for all $x \in P$, $x \neq 0$. Show that if W is any subspace of dimension greater than p there exists a nonzero vector $y \in W$ such that $y^T A y \leq 0$. (Hint: prove that there exists a nonzero vector, $y \in W$, which is orthogonal to all the eigenvectors of A corresponding to positive eigenvalues.)

   iii. Now consider nonsimilarity transformations, $B = S^T A S$ where S is nonsingular. Give an example showing that the spectrum is not preserved by such transformations. Use part (ii.) to prove that B has exactly as many positive, negative and zero eigenvalues as A.

3. (40 pts.) Let $A = A^T$ be a real $d \times d$ symmetric matrix. In the following we derive and analyze the MINRES method for solving $Ax = b$. (Note that actual implementations of MINRES typically involve additional steps to ensure numerical stability, but here you are to consider the algorithm in exact arithmetic.)

i. Define the kth Krylov subspace by:

$$\mathcal{K}_k(A;b) = \text{span}\left\{b, Ab, \ldots, A^{k-1}b\right\}.$$

The Lanczos process is defined by:

$$q_1 = \frac{b}{\|b\|_2}, \quad q_0 = 0, \quad \beta_0 = 0,$$

For $j = 1, \ldots$

$$z = Aq_j, \quad \alpha_j = q_j^T z, \quad z = z - \alpha_j q_j - \beta_{j-1} q_{j-1},$$

$$\beta_j = \|z\|_2, \quad q_{j+1} = \frac{z}{\beta_j}.$$

Prove that in the absence of breakdowns $\beta_j = 0$ the vectors $\{q_1, \ldots, q_k\}$ are an orthonormal basis for the kth Krylov subspace. (Hint: assuming $q_1, \ldots, q_j$ is an orthonormal basis for $\mathcal{K}_j(A;b)$ project z onto the orthogonal complement of $\mathcal{K}_j(A;b)$ as in the Gram-Schmidt algorithm.)

ii. The kth MINRES iterate, x^k, is defined as the solution of the least squares problem:

$$\min_{y \in \mathcal{K}_k} \|Ay - b\|_2.$$

Writing

$$x^k = \sum_{j=1}^{k} c_j^k q_j$$

Show that the expansion coefficients c^k are solutions of the least squares problem:

$$\min_{c \in \mathbb{R}^k} \|T^k c - \|b\|_2 e_1^{k+1}\|_2$$

where T^k is the $(k+1) \times k$ tridiagonal matrix:

$$t_{jj} = \alpha_j, \quad t_{j,j+1} = t_{j+1,j} = \beta_j,$$

and e_1^{k+1} is the first column of the $(k+1) \times (k+1)$ identity matrix.

iii. Briefly outline (no details necessary) an efficient algorithm for computing the iterates, c^k.

iv. Show that if there are no breakdowns MINRES converges in d iterations and if there is a breakdown, $\beta_k = 0$, then $x^k = x$, i.e. the algorithm converges in k steps.