Numerical Analysis, Fall 2009
MS/PhD Qualifying Examination

Write the last four digits of your SSN (not you name) on each work sheet. Complete all problems, providing concise answers with justification.

1. (25 points) Let $M_k^{m\times n}$, $m \geq n$, denote the set of matrices in $\mathbb{C}^{m\times n}$ of rank k. Assume that $A \in M_r^{m\times n}$ and let $B \in M_k^{m\times n}$, $k < r$, be such that

$$\|A - B\|_2 \leq \|A - X\|_2, \quad X \in M_k^{m\times n}.$$

Express B and $\|A - B\|_2$ in terms of the *singular value decomposition* of A:

$$A = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*.$$

2. (25 points) Given a one-parameter family of Hermitian matrices $M(t) \in \mathbb{C}^{n\times n}$, where the coefficients of $M(t)$ are differentiable functions of t, we seek expressions for the variation of the eigenvalues $\{\lambda_1(t), \ldots, \lambda_n(t)\}$ and eigenvectors $\{v_1(t), \ldots, v_n(t)\}$ with respect to t in order to study the behavior of the eigenproblem of a Hermitian matrix under Hermitian perturbation. Show the following. (Assume $\lambda_i(t), v_i(t)$ are differentiable functions of t.)
(a) $dV/dt = VA$, where $V = [v_1, v_2, \ldots, v_n]$ and A is skew-Hermitian.
(b) $d\Lambda/dt = V^*(dM/dt)V - A\Lambda + \Lambda A$, where Λ is the diagonal matrix $(\lambda_1, \ldots, \lambda_n)$.
(c) Use (b) and the fact that A is skew to deduce that $d\lambda_i/dt = v_i^*(dM/dt)v_i$.
(d) Now consider $M(t) = M_0 + tM_1$, where M_0, M_1 are Hermitian and $\|M_1\|_2 = 1$. Show that the eigenvalues of $M(t)$ are stable at $t = 0$ by deriving bounds for $\left|\frac{d\lambda_i}{dt}(0)\right|$.

3. (25 points) Let $A \in \mathbb{C}^{m\times n}$, $m \geq n$, $\text{rank}(A) = n$, and

$$A^\dagger = \int_0^\infty \exp(-tA^*A)A^*dt.$$

Show that $A^\dagger A = I$ and that $AA^\dagger$ is a projection operator. Prove that $A^\dagger$ is a generalized inverse of A.

4. (25 points) Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$ have full rank.
(a) Show that the component x of the solution to the system

$$M\begin{pmatrix} -r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \qquad M = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$$

minimizes $\|Ax - b\|_2$.
(b) Express the condition number of M in terms of the singular values of A.
(c) Write down an explicit expression for M^{-1} in terms of A and A^T.