Numerical Analysis, MS/PhD Qualifying Examination, Fall 2010

Write the last four digits of your Banner ID (not you name) on each work sheet. Please do each of the following five problems, providing concise answers with justification. For problems with multiple parts, each part is assigned equal weight.

1. (20 points) Let $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Consider solving $Ax = b$ using the conjugate gradient method. The kth iterate x_k then satisfies

$$\|x_k - x\|_A \le \|y - x\|_A \text{ for every } y \in x_0 + \mathcal{K}_k(r_0, A),$$

where $\|\cdot\|_A$ denotes the vector A-norm, $r_0 = b - Ax_0$ is the initial residual, and

$$\mathcal{K}_k(r_0, A) = \text{span}\{r_0, Ar_0, \cdots, A^{k-1}r_0\}.$$

Prove that the difference $x_k - x$ satisfies the estimate

$$\|x_k - x\|_A \le \left(\inf_{p\in\mathcal{P}_k} \max_{\lambda\in\sigma(A)} |p(\lambda)|\right) \|x_0 - x\|_A,$$

where $\mathcal{P}_k$ denotes the space of polynomials p with degree $\le k$ such that $p(0) = 1$, and $\sigma(A)$ is the spectrum of A.

2. (20 points) Consider the least squares problem of minimizing

$$r^2(x) = \|b - Ax\|^2,$$

where $A \in \mathbb{R}^{m\times n}$ has rank $n \le m$ and $\|\cdot\|$ is the Euclidean vector norm. Let

$$A = (Q_1, Q_2)\begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR-decomposition of A. Here Q_1, Q_2, and R are respectively $m \times n$, $m \times (m-n)$, and $n \times n$.

(a) Show that the solution of the least squares problem satisfies the QR-equation $Rx = Q_1^T b$ and that the solution is unique. Further show $r(x) = \|Q_2^T b\|$.

(b) Use the QR-equation to show that the least squares solution satisfies the normal equations $(A^T A)x = A^T b$.

3. (20 points) For $b \in \mathbb{R}^n$ and nonsingular $A \in \mathbb{R}^{n\times n}$, consider the following iterative scheme for the solution of $Ax = b$:

$$x_{k+1} = x_k + \alpha(b - Ax_k).$$

(a) Show that if each eigenvalue of A has strictly positive real part, then there exists a real α such that the iteration converges for any starting vector x_0. Discuss how to choose α optimally in the case that A is also symmetric and determine the rate of convergence.

(b) Use the *Gershgorin circle theorem* to find an α that guarantees convergence for the following tridiagonal matrix:

$$A = \begin{pmatrix} 3 & 1 & & & & \\ -1 & 3 & 1 & & & \\ & -1 & 3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 3 & 1 \\ & & & & -1 & 3 \end{pmatrix}.$$

(c) Let $q = \|I - \alpha A\| \leq 1$ for the matrix norm subordinate to a vector norm $\|\cdot\|$. Show that the error can be expressed in terms of the difference between consecutive iterates as

$$\|x - x_{k+1}\| \leq \frac{q}{1-q}\|x_k - x_{k+1}\|.$$

4. (20 points) Let $A \in \mathbb{R}^{n\times n}$ and $\rho(A)$ be the spectral radius of A, that is

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Prove or disprove the following statements.

(a) If $\|\cdot\|$ is a matrix norm induced by a vector norm, then $\rho(A) \leq \|A\|$.

(b) Given any $\epsilon > 0$ there is a vector norm $\|\cdot\|$ and induced matrix norm, also denoted $\|\cdot\|$, such that $\|A\| \leq \rho(A) + \epsilon$.

Hint: Use the following similarity transformation (each matrix is either diagonal or bidiagonal).

$$\begin{pmatrix} \lambda & \epsilon & & & \\ & \lambda & \epsilon & & \\ & & \ddots & \ddots & \\ & & & \lambda & \epsilon \\ & & & & \lambda \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \epsilon^{-1} & & & \\ & & \ddots & & \\ & & & \epsilon^{-k+2} & \\ & & & & \epsilon^{-k+1} \end{pmatrix} \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \epsilon & & & \\ & & \ddots & & \\ & & & \epsilon^{k-2} & \\ & & & & \epsilon^{k-1} \end{pmatrix}$$

5. (20 points) Let B be an $n \times n$ unit upper bidiagonal matrix,

$$B = \begin{pmatrix} 1 & b_1 & & & \\ & 1 & b_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & b_{n-1} \\ & & & & 1 \end{pmatrix}.$$

Derive an algorithm for computing $\kappa_\infty(B)$ exactly (ignoring round off). Your algorithm should be as cheap as possible. Specify the number of additions, multiplications, absolute values, and comparisons it requires.