Numerical Analysis Fall 2015
MS/PhD Qualifying Examination

1. Let $A \in R^{n\times n}$ be an invertible, symmetric positive definite matrix, $b \in R^n$. This problem regards the method of steepest descent to find the solution $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$. Steepest descent is an iterative method that defines a sequence $\mathbf{x}_n$ which converges to the minimizer of the function
$$\phi(x) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}\ .$$
   (a) Let $e(\mathbf{x}) = \mathbf{x} - \mathbf{x}^*$ and $||x||_A = \sqrt{x^T A\mathbf{x}}$. where $\mathbf{x}^* = A^{-1}\mathbf{b}$ solves $A\mathbf{x} = \mathbf{b}$. Prove that $\mathbf{x}$ minimizes $\phi(\mathbf{x})$ if and only if $\mathbf{x}$ minimizes $||e(\mathbf{x})||_A$, and thus $\mathbf{x} = \mathbf{x}^*$, unique.
   (b) Derive a formula for $-\nabla\phi$.
   (c) The vector $-\nabla\phi(\mathbf{x})$ points in the direction of steepest descent of ϕ at $\mathbf{x}$. The method of steepest descent consists of iterating
$$\mathbf{x}_{n+1} = \mathbf{x}_n - \alpha_n \nabla\phi(\mathbf{x}_n)$$
   starting from an initial guess $\mathbf{x}_0$. That is, one steps from $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$ by moving along the direction of steepest descent. Determine the optimal step length α_n that minimizes $\phi(\mathbf{x}_{n+1})$. Explain why the method always converges to the minimizer $\mathbf{x}^*$ of ϕ.
   (d) Write down an algorithm for the full steepest descent iteration. There are three operations inside the main loop.

2. Consider the least squares problem $\min_{\mathbf{x}} ||A\mathbf{x} - \mathbf{b}||_2$ with $A \in R^{m\times n}, m > n$, and $\mathbf{b} \in R^m$.
   (a) Assume A has full rank. Describe how to solve the least squares problem using (i) the normal equations, (ii) the QR factorization, and (iii) the SVD de- composition. Discuss the condition number and the number of operations for each method.
   (b) Discuss what happens when the rank of A is less than n.

3. Determine the rate of convergence of the Rayleigh quotient $\mathbf{r}(v_k) = \mathbf{v}_k^T A\mathbf{v}_k$, to an eigenvalue of $A \in R^{n\times n}, A = A^T$, with vectors $\mathbf{v}_k \in R^n$ given by the normalized power method $\mathbf{v}_{k+1} = A\mathbf{v}_k/||A\mathbf{v}_k||$.

4. Let $f \in C_{[0,1]}$ be given, and u solve
$$-u_{xx} = f\ , \quad x \in [0,1]\ , \quad u(0) = u(1) = 0.$$
   For given N, an approximation v_i of $u(x_i)$, $x_i = ih$, $h = 1/(N+1)$ is given by
$$-v_{i+1} + 2v_i - v_{i-1} = h^2 f_i\ , \quad i = 1, \dots, N\ , \quad h = 1/(N+1), \tag{1a}$$
$$v_0 = v_{N+1} = 0 \tag{1b}$$
   where $f_i = f(x_i)$. Equations (1ab) form a linear system in v_i which can be written as
$$T_N[v_1, \dots, v_N]^T = h^2[f_1, \dots, f_N]^T.$$
   (a) Show that the eigenvalues and eigenvectors of T_N are $\lambda_j = 2(1 - \cos(\frac{\pi j}{N+1}))$ and $\mathbf{z}_j$, with components $z_j^k = \sqrt{\frac{2}{N+1}}\sin(\frac{\pi jk}{N+1})$.

(b) Let $Z = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N]$, and $T_N = Z\Lambda Z^T$, and consider the equivalent problem in 2D,

$$u_{xx} + u_{yy} = f(x, y)$$

where $(x, y) \in D = [0, 1] \times [0, 1]$, and

$$u = 0 \quad \text{on} \quad \partial D \ ,$$

with the approximation $v_{i,j}$ of $u(x_i, y_j)$ given by

$$-v_{i+1,j} - v_{i-1,j} - v_{i,j+1} - v_{i,j-1} + 4v_{i,j} = h^2 f_{i,j} \ , \tag{2a}$$

$$v_{i,0} = v_{i,N+1} = v_{0,j} = v_{N+1,j} = 0 \ , \quad i, j = 1, \dots, N, \tag{2b}$$

where $f_{i,j} = f(x_i, y_j)$, $x_i = ih$, $y_j = jh$, $h = 1/(N+1)$. Show that equations (2ab) can be written as a matrix equation

$$T_N V + V T_N = h^2 F. \tag{3}$$

Here $V_{i,j} = v_{i,j}$ and $F_{i,j} = f_{i,j}$.

(c) Let $V' = Z^T V Z$. Show that the solution V of equation (3) can be found by the following steps

$$F' = Z^T F Z, \tag{4a}$$

$$v'_{j,k} = \frac{h^2 f'_{j,k}}{\lambda_j + \lambda_k}, \tag{4b}$$

$$V = Z V' V^T. \tag{4c}$$

(d) To leading order in N, how many operations does it require to solve (4)?

(e) Outline a method to solve (3) with complexity $O(N^2 \log N)$.

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix}$$

(a) Find the reduced and a full singular value decomposition of A.

(b) Find the best rank-1 approximation B of A in the 2-norm.

(c) Find $||A - B||_2$.