Numerical Analysis Spring 2016
MS/PhD Qualifying Examination

*Instructions: Write your identifying numbers (***not*** your name) on each sheet. Complete all problems. Justify your answers and clearly show all required work.*

1. (a) Let $A = diag\{d_1, d_2, \ldots, d_n\} \in R^{n\times n}$. What are the eigenvalues and eigenvectors of A?
   (b) Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.
   (c) Let $A \in R^{n\times n}$ and let $B = A - \alpha I$ for some scalar α. How do the eigenvalues of A and B compare? Explain.
   (d) Show that all eigenvalues of a nilpotent matrix are 0.

2. Consider the matrix
$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 0 \\ 3 & 6 & 5 \end{pmatrix}$$
   (a) find a nonsingular P such that $P^{-1}AP$ is diagonal.
   (b) Use your result in (a) to write an expression for A^{100} (skip lengthy computation)
   (c) Write an expression for e^A (skip lengthy computation)

3. Consider the least squares problem $\min_{\mathbf{x}} ||A\mathbf{x} - \mathbf{b}||_2$ with $A \in R^{m\times n}, m > n$, and $\mathbf{b} \in R^m$.
   (a) Assume A has full rank. Describe how to solve the least squares problem using (i) the normal equations, (ii) the QR factorization, and (iii) the SVD decomposition. State a reason why you would not use the normal equations.
   (b) Discuss in a few words how the case $rank(A) < n$ differs from the case $rank(A) = n$.

4. Determine the rate of convergence of the Rayleigh quotient $\mathbf{r}(v_k) = \mathbf{v}_k^T A\mathbf{v}_k$, to an eigenvalue of $A \in R^{n\times n}, A = A^T$, with vectors $\mathbf{v}_k \in R^n$ given by the normalized power method $\mathbf{v}_{k+1} = A\mathbf{v}_k/||A\mathbf{v}_k||$, where $\mathbf{v}_0$ is some vector with $||\mathbf{v}_0|| = 1$.

5. Let $x \in R^n$ and $A \in R^{n\times n}$. Let $\rho(A)$ be the spectral radius of A, and $||A||_k$, $k = 1, 2, \infty$ be the matrix norms induced by the corresponding vector norms $||x||_k$.
   (a) Show that $\rho(A) \leq ||A||$ where $||...||$ is any matrix norm induced by a vector norm.
   (b) Show that
$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty \ .$$
   and that the inequalities are sharp (best possible bound for generic x).
   (c) Show that
$$||A||_2 \leq \sqrt{n}||A||_\infty \ .$$
   and that the inequality is sharp.