Numerical Analysis Spring 2018
MS/PhD Qualifying Examination

*Instructions: Write your code number (***not** *your name) on each sheet. Complete all four problems. Clear and concise answers with good justification will improve your score.*

1. Throughout this question A is a real $m \times n$ matrix with full column rank and ϵ refers to machine epsilon.

   (a) Let A have singular values $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n$. Show that the singular values of $A^T A$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

   (b) Define the 2-norm condition number of A (rectangular with full column rank) as the ratio $\kappa_2(A) = \sigma_{\max}(A)/\sigma_{\min}(A)$. Show that $\kappa_2(A^T A) = [\kappa_2(A)]^2$.

   (c) Consider the normal equations $A^T A x = A^T b$ to solve a linear least squares problem. If $A^T A \hat{x} = A^T b + f$ where $\|f\|_2 \leq c\epsilon \|A^T\|_2 \|b\|_2$ then show that,

   $$\frac{\|x - \hat{x}\|_2}{\|x\|_2} \leq c\epsilon \kappa_2(A)^2 \frac{\|A^T\|_2 \|b\|_2}{\|A^T b\|_2}.$$

2. Let $A \in \mathbb{R}^{n \times n}$ such that $A = L + D + U$ where D is the diagonal, L the strictly lower triangular and U the strictly upper triangular part of A. The kth Gauss-Seidel iteration to solve the linear system $Ax = b$ is given by

   $$(D + L)x^{(k)} = -Ux^{(k-1)} + b \tag{1}$$

   where $x^{(k)} \in \mathbb{R}^n$ is the approximate solution after the kth iteration. Let the error after the kth iteration be $e^{(k)} = x^{(k)} - x$ where x is the true solution. Also let

   $$r_i = \sum_{j=1, j \neq i}^{n} |a_{ij}/a_{ii}|, \qquad \text{and} \qquad r = \max_{1 \leq i \leq n} r_i.$$

   (a) Show that the ith component of the error satisfies

   $$e_i^{(k)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(k-1)}$$

   (Hint: Write out equation (1) componentwise.)

   (b) Show that the ith component of the error satisfies,

   $$|e_i^{(k)}| \leq r \|e^{(k-1)}\|_\infty$$

   (Hint: Use induction on the components of the error.)

   (c) Show that the Gauss-Seidel iteration converges to the true solution if A is diagonally dominant.

3. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix.

   (a) Explain how you can compute the QR factorization of A.

   (b) Given any QR factorization, $A = \hat{Q}\hat{R}$ of A, show that there exists a QR factorization, $A = QR$, such that the diagonal entries of R are positive.

   (c) Show that an upper triangular orthogonal matrix, M, must be a diagonal matrix with either 1 or -1 on the diagonal. (Hint: Use induction on the size of the matrix and look at the products MM^T and $M^T M$.)

   (d) Show that the QR factorization, $A = QR$, such that the diagonal entries of R are positive, is unique. (Hint: Assume otherwise and then apply part (c).)

4. Let $\mathbf{v}_0 \in \mathbb{R}^n$ be some vector with $||\mathbf{v}_0|| = 1$. Determine the rate of convergence of the Rayleigh quotient $r(\mathbf{v}_k) = \mathbf{v}_k^\top A \mathbf{v}_k$ to an eigenvalue of $A \in \mathbb{R}^{n \times n}$, $A = A^\top$, with vectors $\mathbf{v}_k \in \mathbb{R}^n$ given by the normalized power iteration method

   $$\mathbf{v}_{k+1} = \frac{A\mathbf{v}_k}{||A\mathbf{v}_k||}.$$

   You can use the fact that an eigenvector of A, say $\mathbf{q}$, is the stationary point of the function $r(\mathbf{v})$, that is $\nabla r(\mathbf{q}) = 0$.