Spring 2019 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please do each of the following five problems, providing concise answers with justification.

1. Given a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, how would you solve the following problems efficiently using Gaussian elimination with partial pivoting?
(a) Solve $A^k x = b$, where $x, b \in \mathbb{R}^n$ and k is a positive integer.
(b) Compute $\alpha = c^T A^{-1} b$, where $c, b \in \mathbb{R}^n$.
(c) Solve $AX = B$, where $X, B \in \mathbb{R}^{n\times m}$.
In each case, describe your algorithm, give pseudocode (e.g. Matlab-type code), assuming that you have a function GEPP that performs Gaussian elimination with partial pivoting (i.e., you do not have to write the code for GEPP). In each case specify the number of flops in your algorithm, including the flops for GEPP.

2. Let $A \in \mathbb{R}^{m\times n}$, with $m > n$, have singular values $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \neq 0$, so that A is full rank. Show the following.
(a) $1 = \|A(A^TA)^{-1}A^T\|_2$
(b) $1 = \|I - A(A^TA)^{-1}A^T\|_2$
(c) $1 = \sigma_n(A)\|(A^TA)^{-1}A^T\|_2$
(d) $1 = \sigma_n(A)^2\|(A^TA)^{-1}\|_2$

3. (a) Let a, b be scalars and A be a square matrix. Prove that if λ is an eigenvalue of A, then $a\lambda + b$ is an eigenvalue of $aA + bI$.
(b) Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix with diagonal entries equal to zero, and $a_{i,i+1} = a_{i+1,i} = 1$, for $i = 1, 2, \ldots, n-1$. For $j = 1, \ldots, n$, let $x^{(j)} \in \mathbb{R}^n$ have ith component $x_i^{(j)} = \sin(ij\pi/(n+1))$. Prove that

$$Ax^{(j)} = 2\cos\left(\frac{j\pi}{n+1}\right)x^{(j)}, \quad j = 1, \ldots, n.$$

(c) Let A be a tridiagonal matrix with $a_{ii} = d$, $a_{i,i+1} = a_{i+1,i} = e$ for all allowable i. Show that the eigenvalues of A consist of the numbers $d + 2e\cos(j\pi/(n+1))$, $j = 1, \ldots, n$.

4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Consider the linear system $Ax = b$ and the following iteration: given $x^0 \in \mathbb{R}^n$, for $k = 0, 1, \ldots$ define

$$x^{k+1} = (I - \tau A)x^k + \tau b.$$

Here I is the identity matrix and τ is a parameter.
(a) State a sufficient condition for the above iteration to converge.
(b) Suppose $m > 0$ is a lower bound and $M > 0$ is an upper bound for the eigenvalues of A (so any eigenvalue λ of A obeys $m \leq \lambda \leq M$). Find the iteration parameter τ (in terms of m and M) which gives rise to the optimal convergence rate when measured in the Euclidean norm. What is the convergence rate?

(c) For the specific n-by-n matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

find lower and upper bounds for the eigenvalues in terms of n. What is the convergence rate when the above scheme is applied to solve $Ax = b$ for this matrix? *Hint:* use explicit expressions for the eigenvalues given by the results of the previous problem.

5. Suppose $0 \neq b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Denote the associated Krylov subspace by $\mathcal{K}_k(A, b) = \operatorname{span}\big(b, Ab, A^2b, \ldots, A^{k-1}b\big)$ for $k \leq n$.

(a) Show that A^{-1} is also symmetric positive definite. This shows that $\|r\|_{A^{-1}} = \sqrt{r^T A^{-1} r}$ defines a norm.

(b) Assume $\mathcal{K}_k$ has dimension k, and consider $Q \in \mathbb{R}^{n\times k}$ with orthonormal columns that span $\mathcal{K}_k$. Show that $T = Q^T AQ \in \mathbb{R}^{k\times k}$ is invertible.

(c) Assuming that the first column of Q is $b/\|b\|_2$, find the unique vector $x \in \mathcal{K}_k$ which obeys the Galerkin condition: $r = b - Ax$ is orthogonal to $\mathcal{K}_k$. *Hint:* start by justifying that x can be expressed as follows: $x = Qy$.

(d) Prove that the Galerkin condition is equivalent to the statement that x minimizes the A^{-1}-norm of the residual, that is minimizes $\|r\|_{A^{-1}}$.